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**Exercises for the lecture**
**High Dimensional Analysis: Random Matrices and Machine Learning**
Summer term 2023
**Sheet 4**
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Besides Wishart matrices the other important random matrix ensemble is given by Wigner matrices. A symmetric matrix $X = X^T \in \mathbb{R}^{n\times n}$ is a *Wigner matrix* if, apart from the symmetry condition, all its entries are independent and identically distributed according to a centred Gaussian distribution (this can be more general, but let us restrict here to Gaussians). In order to have an asymptotic distribution for $n \to \infty$ we have to normalize the entries to have variance $1/n$, i.e., our Wigner matrix has the form

$$X_n = \frac{1}{\sqrt{n}}(x_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n},$$

where

- $x_{ij} \sim N(0,1)$ for all $i, j$,
- $\{x_{ij} : 1 \leq i \leq j \leq n\}$ is independent, and
- $x_{ji} = x_{ij}$ for all $i, j$.

Their asymptotic eigenvalue distribution was determined by Wigner in 1955; this was the first and still most fundamental (asymptotic) result about random matrices. In the following two exercises we will address Wigner's semicircle law from a numerical and a theoretical perspective.

**Exercise 1** (6 points). Generate histograms of the eigenvalues of an $n \times n$ Wigner matrix, where $n \in \{10, 100, 1000, 2000\}$. Do this in each case for at least two realizations, in order to convince yourself that also in this case we have concentration of the eigenvalues around a deterministic asymptotic distribution. This asymptotic distribution is Wigner's semicircle, which has density

$$\psi(t) = \frac{1}{2\pi}\sqrt{4-t^2} \quad \text{on} \quad [-2, 2].$$

Compare your histograms with this semicircle distribution.

***please turn over***

**Exercise 2** (3 + 3 + 3 + 3 points). We will now determine the form of the semicircle in an analytic way relying on the Stieltjes transform, similar as we did it in class for the Marchenko-Pastur distribution. Denote by $S_n$ the Stieltjes transform of our Wigner matrices,

$$S_n(z) = E\left[\operatorname{tr}\big((X_n - zI_n)^{-1}\big)\right]$$

We will try to derive an equation for the limiting Stieltjes transform (assuming that it exists) $S(z) := \lim_{n\to\infty} S_n(z)$, by writing $X_n$ in the form

$$X_n = \frac{1}{\sqrt{n}}\begin{pmatrix} x_{11} & x^T \\ x & Y \end{pmatrix},$$

where $Y \in \mathbb{R}^{(n-1)\times(n-1)}$ contains the last $n-1$ rows and columns of $X_n$ and $x \in \mathbb{R}^{n-1}$ is the vector $x = (x_{21}, \ldots, x_{n1})^T$. The replacement of the Sherman-Morrison formula in this case is given by Schur's complement formula, which says that for a decomposition of $M \in \mathbb{R}^{n\times n}$ in the form

$$M = \begin{pmatrix} a & v^T \\ v & D \end{pmatrix} \qquad D \in \mathbb{R}^{(n-1)\times(n-1)}, \quad v \in \mathbb{R}^{n-1}, \quad a \in \mathbb{R},$$

the inverse of $M$ exists if $D$ is invertible and $a - v^T D^{-1} v \neq 0$, and in this case the $(1,1)$-entry of $M^{-1}$ is given by

$$[M^{-1}]_{11} = \frac{1}{a - v^T D^{-1} v}.$$

(a) Prove the formula above for the $(1,1)$-entry of $M^{-1}$.

*Hint: it might be good to also find formulas for the other entries of* $M^{-1}$.

(b) By applying the formula above to $M = X_n - zI_n$ show that

$$[M^{-1}]_{11} \approx \frac{1}{-z - S_n(z)}.$$

(c) By doing the same with splitting off the $k$-th row and column in $M$, show that the Stieltjes transform of our Wigner matrix satisfies in the limit $n \to \infty$ the equation

$$S(z) = \frac{1}{-z - S(z)}.$$

(d) Solve the equation for $S(z)$ and derive from this, by Stieltjes inversion formula, the formula for the density of the semicircle.

**Exercise 3** (4 + 4 points). Let $Q \in \mathbb{R}^{p\times p}$ and $U, V \in \mathbb{R}^{p\times n}$ be deterministic matrices such that both $Q$ and $Q + UV^T$ are invertible.

(a) Show that $I_n + V^T Q^{-1} U$ is also invertible.

(b) Show that $(Q + UV^T)^{-1} = Q^{-1} - Q^{-1}U(I_n + V^T Q^{-1} U)^{-1} V^T Q^{-1}$.

***please turn over***

**Exercise 4** (3 + 5 + 6 points)**.** Let $p, n \in \mathbb{N}$ with $p$ even and $\gamma := \frac{p}{n}$. In Assignment 2, Exercise 1 we looked on Wishart matrices where $\Sigma$ is not the identity matrix, but has one half of its eigenvalues equal to $t_1 = 1$ and the other half equal to $t_2 = 2$. Let us now consider such a situation with arbitrary $t_1, t_2 \in \mathbb{R}$, i.e., our data matrix is of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{p \times n},$$

where

- the columns of $X \in \mathbb{R}^{\frac{p}{2} \times n}$ are $N(0, t_1 I_{\frac{p}{2}})$-distributed,
- the columns of $Y \in \mathbb{R}^{\frac{p}{2} \times n}$ are $N(0, t_2 I_{\frac{p}{2}})$ distributed, and
- all these column vectors are independent.

Thus the Wishart matrix is of the form

$$\hat{\Sigma} = \frac{1}{n} \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X^T & Y^T \end{pmatrix} = \frac{1}{n} \begin{pmatrix} XX^T & XY^T \\ YX^T & YY^T \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

(a) Recall that, apart from some zeros, $\hat{\Sigma}$ has the same eigenvalues as

$$\check{\Sigma} = \frac{1}{n} \begin{pmatrix} X^T & Y^T \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{n}(X^T X + Y^T Y) \in \mathbb{R}^{n \times n}.$$

Give, for $p \leq n$, the relation between the Stieltjes transforms of $\hat{\Sigma}$ and of $\check{\Sigma}$.

(b) By following the same ideas as in class for the determination of the Marchenko-Pastur law, show that the limit $\check{S}(z)$ of the Stieltjes transform for this $\check{\Sigma}$ satisfies

$$1 + z\check{S}(z) = \frac{\gamma}{2} \frac{t_1 \check{S}(z)}{1 + t_1 \check{S}(z)} + \frac{\gamma}{2} \frac{t_2 \check{S}(z)}{1 + t_2 \check{S}(z)}.$$

(c) If we put $S(z) := \check{S}(z)/\gamma$, then this satisfies the equation

$$S(z) = -\frac{1}{\gamma z} + \frac{1}{2z} \frac{t_1 \gamma S(z)}{1 + t_1 \gamma S(z)} + \frac{1}{2z} \frac{t_2 \gamma S(z)}{1 + t_2 \gamma S(z)}.$$

This $S(z)$ gives us then the density $\psi$ of the asymptotic eigenvalue distribution of $\hat{\Sigma}$ via the Stieljes inversion formula

$$\psi(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im}\big(S(t + i\varepsilon)\big).$$

Let $t_1 = 3$, $t_2 = 15$ and $\gamma = \frac{1}{5}$. In the same diagram, plot the following:

(i) The graph of $\psi$, obtained by numerically applying a fixed-point iteration to calculate $\psi(t) \approx \frac{1}{\pi} \operatorname{Im}\big(S(t + i\varepsilon)\big)$ for $\varepsilon = 0.01$.[1] As a starting point, any point in the complex upper half-plane will work and result in a solution in the complex upper half-plane. Use enough values for $t$ to get a smooth curve!

(Note that there will be an additional pole at 0, coming from the difference between $\hat{\Sigma}$ and $\check{\Sigma}$.)

(ii) A histogram of the eigenvalues of a numerical simulation of the corresponding Wishart matrix with $p = 500$, normalized to fit the density.

[1] Although the equation for $S(z)$ is a cubic one and might thus be solved explicitly, it is easier to solve the equation numerically as a fixed-point equation (especially in more general situations).