10. Operator-valued free convolution via subordination function and the distribution of polynomials in free variables

10.1. Remarks: 1) We want to describe $X_1 + X_2$, for X_1 and X_2 free, in a subordinated form via

$$G_{X_1+X_2}(z) = G_{X_1}(\omega_1(z))$$

$$G_{X_1+X_2}(z) = G_{X_2}(\omega_2(z))$$

for some subordination functions ω_1, ω_2. Let us check, on a formal level, the properties of those (compare also 5.1. of FPT class):

$$\omega_1(z) = G_{X_1}^{\langle -1 \rangle}\left(G_{X_1+X_2}(z)\right)$$

[Note that

$$z \cdot G(z) = 1 + R(G(z)) \cdot G(z)$$

means that (for $z = G^{\langle -1 \rangle}(b)$)

$$G^{\langle -1 \rangle}(b) \cdot b = 1 + R(b) \cdot b, \text{ i.e.}$$

$$G^{\langle -1 \rangle}(b) = b^{-1} + R(b)$$]

Put now $G_1 = G_{x_1}$, $G_2 = G_{x_2}$

$$G = G_{x_1+x_2}$$

same for R

Then we have

$$\omega_1(z) = G_1^{\langle -1\rangle}(G(z))$$

$$= G(z)^{-1} + R_1(G(z))$$

and

$$\omega_2(z) = G(z)^{-1} + R_2(G(z))$$

and thus

$$\omega_1(z) + \omega_2(z) = 2G(z)^{-1} + \underbrace{R_1(G(z)) + R_2(G(z))}$$

$$= R(G(z))$$

$$= z - G(z)^{-1}$$

$$= z + G(z)^{-1}$$

$$= z + G_1(\omega_1(z))^{-1}$$

$$= z + F_1(\omega_1(z))$$

and thus

$$\omega_2(z) = z + \underbrace{F_1(\omega_1(z)) - \omega_1(z)}_{= h_1(\omega_1(z))}$$

where we put

$$F(z) := G(z)^{-1}$$

$$h(z) := F(z) - z = G(z)^{-1} - z$$

So we have

$$\omega_2(z) = z + h_1(\omega_1(z))$$

$$\omega_1(z) = z + h_2(\omega_2(z))$$

$$= z + h_2(z + h_1(\omega_1(z)))$$

The latter is a fixed point equation for $\omega_1(z)$, which can be used for calculating $\omega_1(z)$ via iterations.

2) The crucial point is that the fixed point equation can be used to define $\omega_1(z)$ (and $\omega_2(z)$) not just on some suitably chosen domain, but always on all of $H^+(B)$. To show the convergence of the iterates on all of $H^+(B)$ one uses again the Earle-Hamilton Theorem.

3) To make the formal calculations above rigorous is much harder than in the scalar-valued case (in particular, as the involved domains are harder to control), but it can be done

10.2. Theorem (Belinschi, Mai, Speicher 2013):

Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator-valued C^*-probability space and consider selfadjoint $X_1, X_2 \in \mathcal{A}$ which are free w.r.t. E. Then there exists a unique pair of Fréchet analytic maps $\omega_1, \omega_2 : H^+(\mathcal{B}) \to H^+(\mathcal{B})$ so that

(i) $\operatorname{Im} \omega_j(z) \geq \operatorname{Im} z \qquad \forall z \in H^+(\mathcal{B}),\ j = 1,2$

(ii) $F_1(\omega_1(z)) + z = F_2(\omega_2(z)) + z = \omega_1(z) + \omega_2(z)$
$$\forall z \in H^+(\mathcal{B})$$

(iii) $G_1(\omega_1(z)) = G_2(\omega_2(z)) = G(z)$
$$\forall z \in H^+(\mathcal{B})$$

Moreover, if $z \in H^+(\mathcal{B})$, then $\omega_1(z)$ is the unique fixed point of the map

$f_z : H^+(\mathcal{B}) \to H^+(\mathcal{B})$

$$f_z(w) = h_2(h_1(w) + z) + z,$$

and $\omega_1(z) = \lim_{n\to\infty} f_z^{n}(w)$ for any $w \in H^+(\mathcal{B})$.

Same statements hold for ω_2, where f_z is replaced by $w \mapsto h_1(h_2(w) + z) + z$

<u>10.3. Remark:</u> This can then be used together with the linearization to compute numerically distributions of polynomials in free variables. This has relevance for the asymptotic eigenvalue distribution of random matrices. Assume that $X_N^{(1)}, \dots, X_N^{(d)}$ are $N \times N$ random matrices which are asymptotically free, i.e.

$$(X_N^{(1)}, \dots, X_N^{(d)}) \xrightarrow{N \to \infty} (X_1, \dots, X_d)$$

where $X_1, \dots, X_d$ are free

Then, for any polynomial $p \in \mathbb{C}\langle x_1, \dots, x_d \rangle$,

$$p(X_N^{(1)}, \dots, X_N^{(d)}) \xrightarrow{N \to \infty} p(X_1, \dots, X_d)$$

the distribution of this can be calculated via linearization and operator-valued free convolution

Note the following typical situations for asymptotically free random matrices:

i) independent GUE are asympt. free

ii) GUE are asympt free from deterministic (e.g. diagonal) matrices

iii) "randomly rotated" matrices are asymptotically free:
for $D_1^{(N)}, D_2^{(N)}$ deterministic (e.g. diagonal) matrices and U_N Haar unitary $N \times N$ random matrices, we have that
$D_1^{(N)}$ and $U_N D_2^{(N)} U_N^*$ are asymptotically free

So in particular, asymptotically the eigenvalue distribution of
$p(D_1^{(N)}, U_N D_2^{(N)} U_N^*)$ is given by the distribution of $p(X_1, X_2)$, where
X_1, X_2 are free and

$$\mu_{D_1^{(N)}} \to \mu_{X_1}$$

$$\mu_{D_2^{(N)}} \to \mu_{X_2}$$