Distribution of Polynomials in Free Variables

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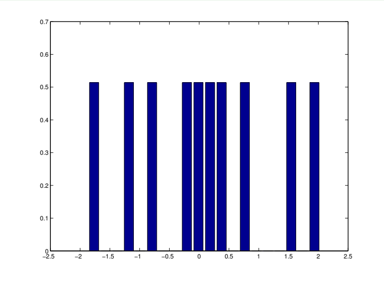
Wigner and Voiculescu

Wigner's semicircle law (Wigner 1955)

10 eigenvalues of GUE(10)



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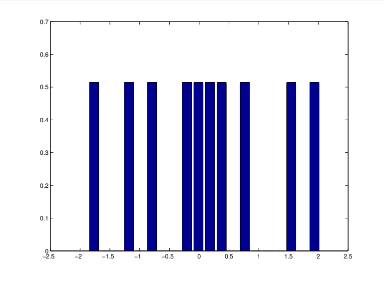

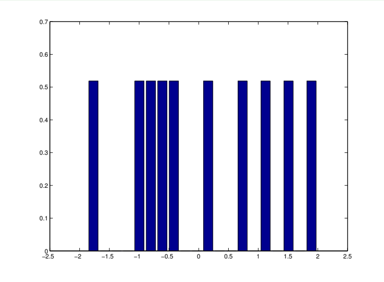

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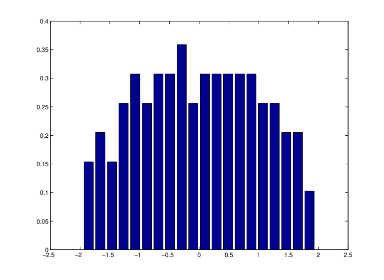
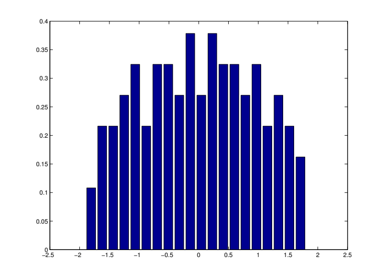
100 eigenvalues of GUE(100)





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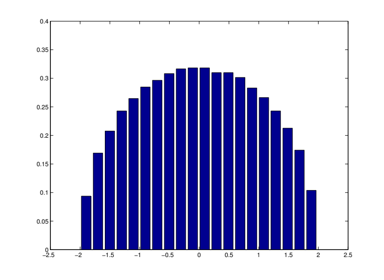
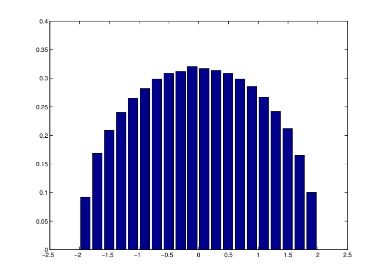
3000 eigenvalues of GUE(3000)





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- GUE is asymptotically free from deterministic (e.g., diagonal) matrix
- "randomly rotated" matrices are asymptotically free:
  - let $D_1^{(N)}$, $D_2^{(N)}$ be deterministic (e.g., diagonal) matrices
  - and U_N a Haar unitary random matrix,

  then

$$X_N = U_N D_1^{(N)} U_N^* \qquad \text{and} \qquad Y_N = D_2^{(N)}$$

  are asymptotically free

Calculation of asymptotic eigenvalue distribution of polynomials

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$$(X_N, Y_N) \to (X, Y)$$

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- The distribution of $p(X, Y)$ is given by the Cauchy transform of a linearization

$$\hat{p}(X, Y) = b_0 \otimes 1 + b_1 \otimes X + b_2 \otimes Y$$

via

$$G_{p(X,Y)}(z) = [G_{\hat{p}(X,Y)}(\Lambda(z))]_{1,1}$$

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- and the use of the subordination calculation of the latter as the sum of two free random variables

Distribution of $p(x, y)$ for x and y free

Calculation of distribution of p by linearization and operator-valued convolution (Belinschi, Mai, Speicher 2013)

- we want the distribution of $p(x, y) = xy + yx + x^2$

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$$\hat{p}(x, y) = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

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$$\hat{p}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y$$

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- for this we have analytic theory of operator-valued free convolution
- this allows to calculate the operator-valued Cauchy transform of $\hat{p}$, and thus the Cauchy transform of p

How to calculate operator-valued free convolution

Theorem (Belinschi, Mai, Speicher 2013)

Consider

$$\hat{p}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y =: X + Y$$

How to calculate operator-valued free convolution

Theorem (Belinschi, Mai, Speicher 2013)

Consider $\hat{p} = X + Y$ Then X and Y are free in the operator-valued sense and there exists a unique pair of (Fréchet-)holomorphic maps $\omega_1, \omega_2 : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$, such that

$$G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$$

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$$G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$$

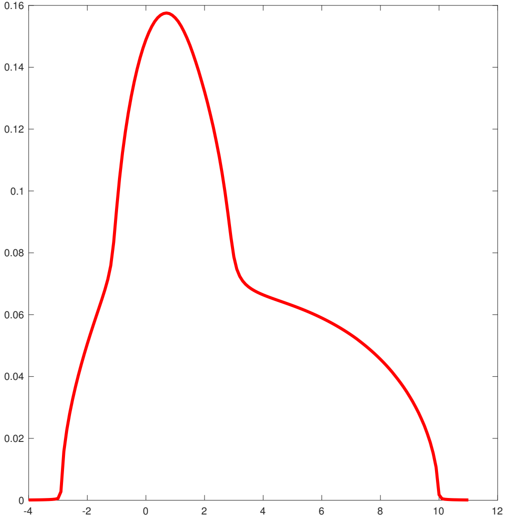
Moreover, ω_1 and ω_2 can easily be calculated via the following fixed point iterations on $\mathbb{H}^+(\mathcal{B})$

$$w \mapsto h_Y(b + h_X(w)) + b \qquad \text{for } \omega_1(b)$$
$$w \mapsto h_X(b + h_Y(w)) + b \qquad \text{for } \omega_2(b)$$

where we put $h_X(b) := G_X(b)^{-1} - b$ and $h_Y(b) := G_Y(b)^{-1} - b$;

$$\mathcal{B} = M_3(\mathbb{C}), \qquad \mathbb{H}(\mathcal{B})^+ := \{b \in \mathcal{B} \mid \Im b = \frac{b - b^*}{2i} > 0\}$$

polynomial $p(x, y) = xy + yx + x^2$



- $p(X, S)$
  $\mu_X = \frac{1}{4}(2\delta_{-2} + \delta_{-1} + \delta_{+1})$
  $\mu_S =$ semicircle

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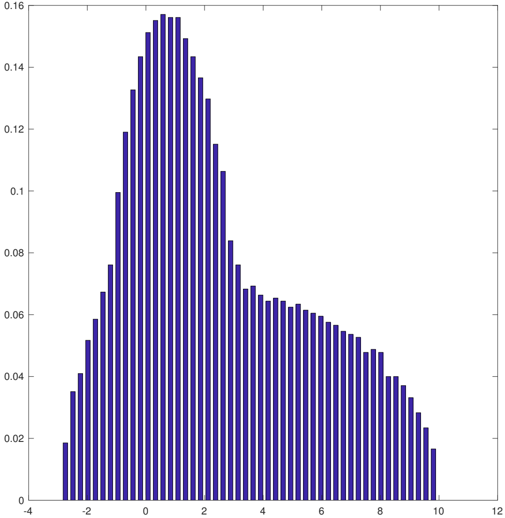

- $p(X_N, A_N)$
  $X_N = \text{diag}(-2, -2, -1, 1)$
  A_N GUE(N)
  N=4000

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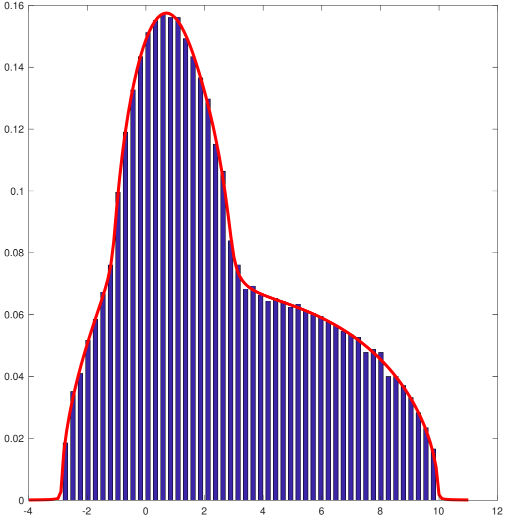

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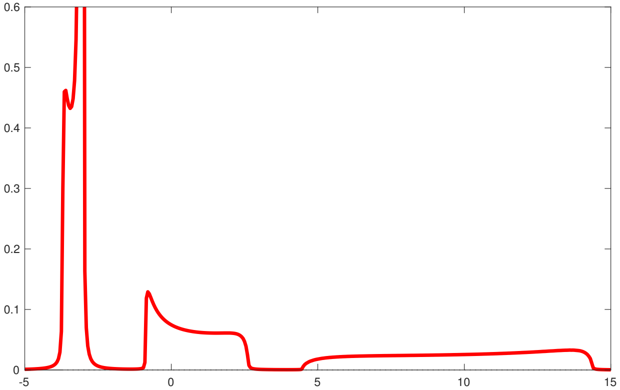

- $p(X,Y)$
  $\mu_X = \frac{1}{2}(\delta_1 + \delta_3)$
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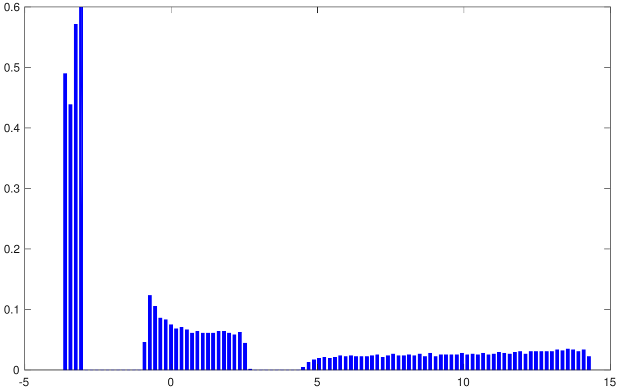

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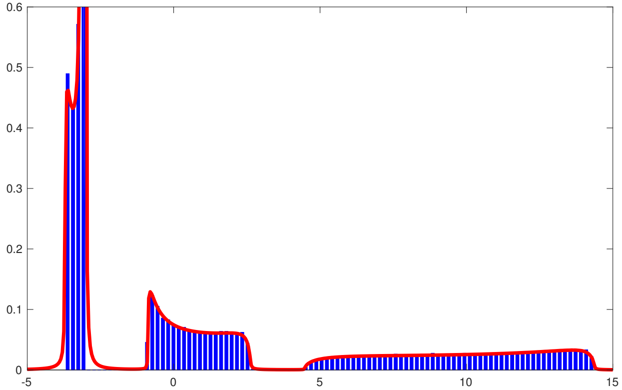

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