7. Matrices of semicirculars and matrix-valued semicirculars (and block random matrices)

7.1. Remark: In 6.3. we have seen that matrices of free semicirculars are matrix-valued semicirculars. We restrict here to the special case where $\mathcal{B} = \mathbb{C}$, i.e. the entries of our matrices are scalar-valued free semicirculars. Let us first give the precise statement for this.

7.2. Proposition: Let $(\mathcal{A}, \varphi)$ be a C^*-probability space and $s_1, \dots, s_d$ be free standard semicirculars (i.e. $\varphi(s_i^2) = 1$). For $n \geq 1$ and selfadjoint $b_1, \dots, b_d \in M_n(\mathbb{C})$ we consider

$$S := b_1 \otimes s_1 + \dots + b_d \otimes s_d \in M_n(\mathbb{C}) \otimes \mathcal{A}$$
$$\hat{=} M_n(\mathcal{A})$$

Then S is in the matrix-valued C^*-probability space $(M_n(\mathcal{A}), M_n(\mathbb{C}), id \otimes \varphi)$ a matrix-valued operator-valued semicircular element with covariance $\eta: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $\eta(b) = \sum_{i=1}^{d} b_i b b_i$

Proof: Assignment

7.2. Remarks: 1) We are now, however, interested in S as a scalar-valued random variable in the C^*-probability space $(M_n(\mathcal{A}), tr \otimes \varphi)$ i.e. instead of the operator-valued Cauchy transform

$$G_S : H^+(M_n(\mathbb{C})) \to H^-(M_n(\mathbb{C}))$$
$$b \mapsto id \otimes \varphi[(b-S)^{-1}]$$

we need the scalar-valued Cauchy transform

$$g_S : H^+(\mathbb{C}) \to H^-(\mathbb{C})$$
$$z \mapsto tr \otimes \varphi[(z-S)^{-1}]$$

Note that for $z \in \mathbb{C}$ we clearly have

$$g_S(z) = tr[G_S(z \cdot 1)]$$

So if we can calculate G_S, we can from this also get g_S.

2) Note that being semicircular on an operator-valued level does in general not imply to be semicircular on a scalar-valued level.

7.3. Example: Consider, for $\alpha, \beta \in \mathbb{R}$,

$$S = \begin{pmatrix} \alpha s_1 & 0 \\ 0 & \beta s_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \otimes s_1 + \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \otimes s_2$$

Then S is for all α, β an $M_2(\mathbb{C})$-valued semicircular element. However, on the scalar level we have the second moment

$$\mathrm{tr} \otimes \varphi[S^2] = \frac{1}{2}(\alpha^2 \varphi(s_1^2) + \beta^2 \varphi(s_2^2)) = \frac{1}{2}(\alpha^2 + \beta^2)$$

if S is semicircular, then its fourth moment must be given by

$$\mathrm{tr} \otimes \varphi[S^4] = 2 \cdot (\mathrm{tr} \otimes \varphi[S^2])^2$$

$\Vert$ (⊔⊔ and ⊔͡⊔ give same contr.)

$$\frac{1}{2}(\alpha^4 \varphi(s_1^4) + \beta^4 \varphi(s_2^4)) = \alpha^4 + \beta^4$$

But $\alpha^4 + \beta^4 = \frac{1}{2}(\alpha^2 + \beta^2)^2$ only if $\alpha = \beta$

Thus in general, semicircularity is not preserved, but there are special cases where it is.

7.4. Theorem: Consider unital C^*-algebras $\mathcal{D} \subset \mathcal{B} \subset \mathcal{A}$ with conditional expectations $E_\mathcal{B}: \mathcal{A} \to \mathcal{B}$ and $E_\mathcal{D}: \mathcal{A} \to \mathcal{D}$, which are compatible in the sense that

$$E_\mathcal{D} \circ E_\mathcal{B} = E_\mathcal{D}.$$

Consider a $\mathcal{B}$-valued semicircular element $S \in \mathcal{A}$, with covariance $\eta : \mathcal{B} \to \mathcal{B}$ with $\eta(b) = E_{\mathcal{B}}[SbS]$.

If $\eta(\mathcal{D}) \subset \mathcal{D}$, then S is also a $\mathcal{D}$-valued semicircular element, with covariance given by the restriction of η to $\mathcal{D}$.

<u>7.5 Example</u>: Reconsider Example 7.3; there $\mathcal{D} = \mathbb{C}$, $\mathcal{B} = M_2(\mathbb{C})$, $E_{\mathcal{D}} = \varphi$, $E_{\mathcal{B}} = id \otimes \varphi$

$\eta : \mathcal{B} \to \mathcal{B}$ is given by

$$\eta \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = id \otimes \varphi \begin{pmatrix} b_{11}\alpha^2 s_1^2 & b_{12}\alpha\beta s_1 s_2 \\ b_{21}\beta\alpha s_2 s_1 & b_{22}\beta^2 s_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2 b_{11} & 0 \\ 0 & \beta^2 b_{22} \end{pmatrix}$$

To check that $\eta(\mathcal{D}) \subset \mathcal{D}$ we just have to see that $\eta(1) \in \mathbb{C}$; but

$$\eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} \in \mathbb{C} \cdot 1 \quad \text{if } \alpha = \beta$$

[one might note that η maps always into diagonal matrices$^{\mathcal{D}}$, and thus in this case S is always a $\mathcal{D}$-valued semicircular!]

Proof of 7.4: We have the Cauchy transforms

$$G(b) = E_B[(b-S)^{-1}] \quad \text{for } b \in H^+(B)$$

and

$$g(d) = E_D[(d-S)^{-1}] \quad \text{for } d \in H^+(D)$$

Note that $H^+(D) \subset H^+(B)$ and

$$g(d) = E_D \underbrace{E_B[(d-S)^{-1}]}_{G(d)}$$

$$= E_D[G(d)]$$

The main claim is to see that

$$(*) \quad G(d) \in D \qquad \forall d \in H^+(D);$$

then we have that

$$g(d) = G(d) \qquad \forall d \in H^+(D)$$

and the equation

$$b \cdot G(b) = 1 + \eta(G(b)) \cdot G(b) \quad (b \in H^+(B)$$

gives for $b = d \in H^+(D)$:

$$d \cdot g(d) = 1 + \eta(G(d)) \cdot G(d)$$

which shows that g is the Cauchy transform of a D-valued semicircular element with covariance $\eta|D$.

Proof of (*): We know, by 6.8, that we get $G(d) \in H^-(B)$ as the limit of the iterates $w_n = \mathcal{F}_d^n(w_0)$ for arbitrary $w_0 \in H^-(B)$, with

$$\mathcal{F}_d(w) = [d - \eta(w)]^{-1}$$

Now note that since η maps D to D the map $\mathcal{F}_d$ also maps D to D; hence if we also choose $w_0 \in H^-(D) \subset H^-(B)$ (as we are free to do), all iterates w_n, and thus also their limit $G(d)$, are in D. □

7.6. Remark: Note the relevance of this for random matrices. If $X_1^{(N)}, \dots, X_d^{(N)}$ are independent Gaussian $N \times N$ random matrices, then we know (see FPT that $(X_1^{(N)}, \dots, X_d^{(N)}) \xrightarrow{N \to \infty} (S_1, \dots, S_d)$ in distribution. But this implies that

$$\underbrace{b_1 \otimes X_1^{(N)} + \dots + b_d \otimes X_d^{(N)}}_{\substack{nN \times nN \text{ block random} \\ \text{matrices, considered as} \\ \text{scalar-valued random variables}}} \xrightarrow{N \to \infty} S = b_1 \otimes S_1 + \dots + b_d \otimes S_d$$

in distribution w.r.t. $tr_n \otimes tr_N$, $tr_n \otimes \varphi$