INTRODUCTION

Avitzour [Avi] and Voiculescu [Voi1] introduced around 1982, following earlier investigations of Ching [Chi], the notion of a reduced free product of C^*-algebras. This construction was meant as a replacement of the concept 'tensor product' in order to get another possibility for constructing new C^*-algebras from given ones. The free product was modelled according to the familiar notion of a free product of groups. It was Voiculescu who realized that this concept of a free product of C^*-algebras is not only important as a technical tool in the structure theory of C^*-algebras, but that it deserves attention on its own. In particular, he recognized that the concept of freeness is quite analogous to the classical probabilistic notion of independence and thus leads canonically to the notion of 'free convolution'. In a series of papers [Voi2,Voi3,Voi5,BV1,BV2] (see also [VDN]) he developed the free analogues of many classical probabilistic concepts. The main tool for his investigations on the free convolution is the 'R-transform', which replaces the logarithm of the classical Fourier transform.

In our own work we tried to understand the results of Voiculescu from a more combinatorial point of view, in the tradition of the algebraic approach of von Waldenfels [GvW] to central limit theorems. In the course of our investigations it turned out that the structure of the free convolution and, more generally, of the free product is governed by the lattice of non-crossing partitions. These non-crossing partitions were introduced in 1972 by Kreweras [Kre], but up to our investigations they were only examined from a purely combinatorial point of view and no connection with probabilistic notions has been made. The first appearing of non-crossing partitions in the quantum probabilistic context was in our proof of the free central and Poisson limit theorems [Spe1]. Inspired by Rota's combinatorial point of view on classical probability theory [Rot1,Rot2], we could finally describe the free convolution and the free product in terms of multiplicative functions on the lattice of non-crossing partitions [Spe4]. This again shows the complete analogy of freeness with independence, since the latter can be described in the same way by multiplicative functions on the lattice of all partitions.

Our work centers around the notion of 'non-crossing' cumulants, which are calculated in a very specific way from moments with the help of the lattice of non-crossing partitions. These cumulants have the characterizing property that they linearize the free convolution. Abstractly, these quantities appeared also in Voiculescu's investigations on the free convolution, but only their concrete description with the help of non-crossing partitions clarifies their structure.

Our approach to the free convolution and free product gives a complementary point of view of the more operator algebraic investigations of Voiculescu. In particular, the structure of the R-transform becomes more transparent and the main formula for the R-transform can be understood as a translation of obvious combinatorial statements about non-crossing partitions to generating power series.

Furthermore, as an extra benefit, our description allows the generalization of Voiculescu's results for one random variable to the case of m non-commuting random variables for arbitrary m [Nic1] and, what is more important, to stochastic processes. In this context one can deal with stochastic differential equations for some quantity $U(t)$ and we could show [NSp1] that the generalization to this frame of Voiculescu's main formula for the R-transform is nothing but an expansion formula or generalized master equation for the description of the evolution of the average $\langle U(t)\rangle$, i.e. for the description of a projected dynamics. Up to that moment, essentially two different forms of such expansion formulas were known, each connected with some special form of cumulants, namely with the so-called ordered and the partial cumulants. Thus, starting from the notion of free product, we have been led to the introduction of a new form of cumulants, the non-crossing ones, which show their usefulness and power in providing a new possibility for describing a projected dynamics.

In this context of generalized master equations it becomes evident at once that the scalar-valued theory, as described up to this moment, is too restrictive and should be generalized to the operator-valued case. The reason is that in physical applications one usually does not project to scalars by taking a trace, rather one projects to the algebra B of some 'small' system by taking a partial trace. Mathematically this amounts to working with conditional expectations instead of states and to replacing free products by free products with amalgamation over an operator algebra B.

Motivated by mathematical reasons, Voiculescu introduced from the very beginning such a generalization of the free product to the operator-valued case [Voi1] and extended also some of his probabilistic investigations to this frame [Voi4].

Here, we want to generalize our combinatorial description of the free product to the operator-valued case. It will turn out that the lattice of non-crossing partitions can again be used for such a description. Indeed, it is this general frame where our combinatorial approach shows its full power and beauty. In particular, the definition of the non-crossing cumulants becomes most natural in this frame and, compared with the ordered and partial cumulants, the non-crossing cumulants are the only ones which behave nicely in all respects, a fact which cannot be seen clearly in the scalar-valued case.

Finally, we want to point out that there exist also other interesting investigations in the context of free probability theory which we will not address in the following, namely free convolution of measures with unbounded support [Maa,BV2], q-deformation of free convolution [Nic2,Nic3], possible occurence of Wigner's semicircle law in quantum electrodynamics [AL], results on quasitraces on C^*-algebras [Haa], the relation between freeness and random matrices [Voi3,VDN,Bia1,Bia2, NSp1,Spe3], and, most spectacular, applications to free group factors [Dyk1,Dyk2, Rad1,Rad2] and subfactors [Pop,Rad2,Boc].