CHAPTER IV

OPERATOR-VALUED FREE PROBABILITY THEORY

In this chapter, we view the elements $a \in A$ as B-valued random variables. This leads canonically to the operator-valued free analogues of many classical probabilistic concepts, in particular to the notion of 'free convolution'. Our main emphasis will be on the notion of cumulants, which linearize the free convolution. The basic definitions and ideas are inspired by [Voi4], where the case of one random variable is treated. We generalize this to m (non-commuting) random variables for arbitrary m. New are our concept of compound B-Poisson distributions and our investigations around the positivity of the appearing distributions and on infinitely divisible distributions for the operator-valued case.

For the classical case we refer to [Fel,Shi], for the quantum probabilistic context to [AFL,CH,GvW,Heg,Küm1,Mey,Par,VDN].

In the following, B will be a fixed unital algebra. If we are interested in the positivity of the appearing distributions, then we shall always assume B to be a C^*-algebra.

4.1. B-valued random variables and free convolution

Let us start by defining the basic notions of a non-commutative probability theory, namely probability space, distributions, and free convolution. Since the latter is only a special case of the amalgamated free product, our investigations of the foregoing chapter give us the concept of cumulants as the main tool for describing the free convolution.

4.1.1. Definition. 1) A pair (A, φ), consisting of an algebra A over B and a B-functional $\varphi : A \to B$, is called a *probability space (over B)*.
2) Given a probability space (A, φ) over B, we call an element $a \in A$ a *B-valued random variable*. Such a situation will be denoted by $a \in (A, \varphi)$.

As in the classical case, all probabilistic information about a random variable is contained in its distribution.

4.1.2. Definition. 1) Let $B\langle X\rangle := \mathbb{C}\langle X\rangle *_{\mathbb{C}} B$ be the algebra generated freely by B and an indeterminate X. For a probability space (A, φ) over B and a random variable $a \in (A, \varphi)$ we define the B-functional

$$\nu_a : B\langle X\rangle \to B \qquad \text{by} \qquad \nu_a := \varphi \circ \tau_a,$$

where $\tau_a : B\langle X\rangle \to A$ is the unique homomorphism such that $\tau_a(b) = b$ for all $b \in B$ and $\tau_a(X) = a$. This ν_a is called the *distribution* of the random variable a. The set of all possible distributions will be denoted by Σ_B,

$$\Sigma_B := \{\nu : B\langle X\rangle \to B \mid \nu\ B\text{-functional}\}.$$

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2) More generally, for $m \in \mathbb{N}$ let

$$B\langle X_1, \ldots, X_m\rangle := B\langle X_1\rangle *_B \cdots *_B B\langle X_m\rangle$$

be the algebra generated freely by B and m non-commuting indeterminates X_1, $\ldots, X_m$. For a probability space (A, φ) over B and m random variables $a_1, \ldots, a_m \in (A, \varphi)$, we define the B-functional

$$\nu_{(a_1,\ldots,a_m)} : B\langle X_1, \ldots, X_m\rangle \to B \qquad \text{by} \qquad \nu_{(a_1,\ldots,a_m)} := \varphi \circ \tau_{(a_1,\ldots,a_m)},$$

where $\tau_{(a_1,\ldots,a_m)} : B\langle X_1, \ldots, X_m\rangle \to A$ is the unique homomorphism such that $\tau_{(a_1,\ldots,a_m)}(b) = b$ for all $b \in B$ and $\tau_{(a_1,\ldots,a_m)}(X_i) = a_i$ for all $i = 1, \ldots, m$. This $\nu_{(a_1,\ldots,a_m)}$ is called the *(joint) distribution* of the random variables $a_1, \ldots, a_m$. The set of all possible joint distributions of m B-valued random variables will be denoted by $\Sigma_B^{(m)}$,

$$\Sigma_B^{(m)} := \{\nu : B\langle X_1, \ldots, X_m\rangle \to B \mid \nu \ B\text{-functional}\}.$$

In particular, $\Sigma_B^{(1)} = \Sigma_B$.

In classical probability theory, the considered distributions are usually states, i.e. positive functionals. Thus we should put our main emphasis also on the corresponding non-commutative generalization.

4.1.3. Notation. If B is a C^*-algebra, then we equip $B\langle X_1, \ldots, X_m\rangle$ with the canonical $*$-structure given by $X_i^* = X_i$ for all $i = 1, \ldots, m$ and denote the set of positive distributions by

$$\Sigma_B^+ := \{\nu \in \Sigma_B \mid \nu \text{ positive}\}$$

and

$$\Sigma_B^{(m)+} := \{\nu \in \Sigma_B^{(m)} \mid \nu \text{ positive}\}.$$

As the classical convolution describes the distribution of the sum of independent random variables, the free convolution does the same for random variables which are free.

4.1.4. Definition. The *free convolution* of distributions

$$\begin{aligned} \boxplus : \Sigma_B^{(m)} \times \Sigma_B^{(m)} &\to \Sigma_B^{(m)} \\ (\nu_1, \nu_2) &\mapsto \nu_1 \boxplus \nu_2 \end{aligned}$$

is defined as follows (for arbitrary $m \in \mathbb{N}$): If $\nu_1 = \nu_{(a_1,\ldots,a_m)}$ and $\nu_2 = \nu_{(\tilde{a}_1,\ldots,\tilde{a}_m)}$ are the joint distributions of B-valued random variables $a_1, \ldots, a_m \in (A, \varphi)$ and $\tilde{a}_1, \ldots, \tilde{a}_m \in (A, \varphi)$, respectively, for some probability space (A, φ) and if $\{a_1, \ldots, a_m\}$ and $\{\tilde{a}_1, \ldots, \tilde{a}_m\}$ are free in (A, φ), then $\nu_{(a_1+\tilde{a}_1,\ldots,a_m+\tilde{a}_m)}$, the joint distribution of $a_1 + \tilde{a}_1, \ldots, a_m + \tilde{a}_m \in (A, \varphi)$, depends only on the distributions ν_1 and ν_2 and is called the free convolution of ν_1 and ν_2, denoted by

$$\nu_1 \boxplus \nu_2 := \nu_{(a_1+\tilde{a}_1,\ldots,a_m+\tilde{a}_m)}.$$

Note that, given $\nu_1, \nu_2 \in \Sigma_B^{(m)}$, it is always possible to find realizations $a_1, \dots, a_m, \tilde{a}_1, \dots, \tilde{a}_m \in (A, \varphi)$ as required in the definition. Namely, we can take

$$(A, \varphi) = (B\langle X_1, \dots, X_m\rangle *_B B\langle \tilde{X}_1, \dots, \tilde{X}_m\rangle, \nu_1 * \nu_2),$$

where

$$\nu_1 : B\langle X_1, \dots, X_m\rangle \to B, \qquad \nu_2 : B\langle \tilde{X}_1, \dots, \tilde{X}_m\rangle \to B,$$

and

$$a_i := X_i, \qquad \tilde{a}_i := \tilde{X}_i$$

for $i = 1, \dots, m$.

Our general theory for the amalgamated free product can now be specified for the description of the free convolution. In particular, Theorem 3.5.6 ensures that the free convolution preserves the positivity of the distributions.

4.1.5. Corollary. *The free convolution of positive distributions is positive, too, i.e. (for B a C^*-algebra)*

$$\boxplus : \Sigma_B^+ \times \Sigma_B^+ \to \Sigma_B^+ \qquad \textit{and} \qquad \boxplus : \Sigma_B^{(m)+} \times \Sigma_B^{(m)+} \to \Sigma_B^{(m)+}.$$

For arbitrary $m \in \mathbb{N}$, the computation of $\nu_1 \boxplus \nu_2$ out of ν_1 and ν_2 is done in the same way as for the free product. Namely, for $\nu \in \Sigma_B^{(m)}$, $\nu : B\langle X_1, \dots, X_m\rangle \to B$, we define its moment function $\hat{\varphi}_\nu = (\varphi_\nu^{(n)}) \in \mathbf{I}^m(B\langle X_1, \dots, X_m\rangle, B)$ by

$$\varphi_\nu^{(n)} : \underbrace{B\langle X_1, \dots, X_m\rangle \otimes_B \cdots \otimes_B B\langle X_1, \dots, X_m\rangle}_{n\text{-times}} \to B$$

and

$$\varphi_\nu^{(n)}(a_1 \otimes \cdots \otimes a_n) := \nu(a_1 \dots a_n) \qquad \text{for all } a_1, \dots, a_n \in B\langle X_1, \dots, X_m\rangle.$$

¿From this one can calculate the corresponding cumulant function

$$\hat{c}_\nu = (c_\nu^{(n)}) := \hat{\varphi}_\nu \star \mu \in \mathbf{I}^c(B\langle X_1, \dots, X_m\rangle, B).$$

4.1.6. Notation. All information is contained in the collection

$$\xi(\nu) := (\xi^{(\nu)}_{n;i_0,\dots,i_n})_{n\in\mathbb{N}_0; i_0,\dots,i_n \in \{1,\dots,m\}}$$

of linear mappings, defined by (for $n \in \mathbb{N}_0$ and $i_0, \dots, i_n \in \{1, \dots, m\}$)

$$\xi^{(\nu)}_{n;i_0,\dots,i_n} : \underbrace{B \times \cdots \times B}_{n\text{-times}} \to B$$

with $(b_1, \dots, b_n \in B)$

$$\begin{aligned} \xi^{(\nu)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) :&= c_\nu^{(n+1)}(X_{i_0} \otimes b_1 X_{i_1} \otimes b_2 X_{i_2} \otimes \cdots \otimes b_n X_{i_n}) \\ &= c_\nu^{(n+1)}(X_{i_0} b_1 \otimes X_{i_1} b_2 \otimes \cdots \otimes X_{i_{n-1}} b_n \otimes X_{i_n}). \end{aligned}$$

In particular,

$$\xi^{(\nu)}_{0;i_0} : \mathbb{C} \to B, \qquad \xi^{(\nu)}_{0;i_0}(1) = c^{(1)}_\nu(X_{i_0}) = \nu(X_{i_0}).$$

Let us denote by $B\langle X_1, \dots, X_m\rangle_0$ the B-B-bimodule (we use the notation $\mathcal{X}_m := \operatorname{span}\{X_1, \dots, X_m\}$)

$$\begin{aligned} B\langle X_1, \dots, X_m\rangle_0 :&= B\mathcal{X}_m B \oplus B\mathcal{X}_m B\mathcal{X}_m B \oplus B\mathcal{X}_m B\mathcal{X}_m B\mathcal{X}_m B \oplus \dots \\ &\subset B\langle X_1, \dots, X_m\rangle, \end{aligned}$$

i.e. those polynomials in $B\langle X_1, \dots, X_m\rangle$ which have no constant term. Then we will consider $\xi(\nu)$ also as a bimodule map from $B\langle X_1, \dots, X_m\rangle_0$ to B, i.e.

$$\xi(\nu) : B\langle X_1, \dots, X_m\rangle_0 \to B$$

with ($n \in \mathbb{N}_0$, $i_0, \dots, i_n \in \{1, \dots, m\}$, $b_0, \dots, b_{n+1} \in B$)

$$\begin{aligned} \xi(\nu)(b_0 X_{i_0} b_1 X_{i_1} \dots b_n X_{i_n} b_{n+1}) &= b_0\big(\xi^{(\nu)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n)\big) b_{n+1} \\ &= c^{(n+1)}_\nu(b_0 X_{i_0} \otimes b_1 X_{i_1} \otimes \dots \otimes b_n X_{i_n} b_{n+1}). \end{aligned}$$

If $\nu = \nu_{(a_1,\dots,a_m)} \in \Sigma^{(m)}_B$ is the joint distribution of random variables $a_1, \dots, a_m \in (A, \varphi)$, then we will also write

$$\xi(a_1, \dots, a_m) = \xi(\nu) \qquad \text{and} \qquad \xi^{(a_1,\dots,a_m)}_{n;i_0,\dots,i_n} = \xi^{(\nu)}_{n;i_0,\dots,i_n}.$$

The latter is of course nothing else but

$$\xi^{(a_1,\dots,a_m)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) = c^{(n+1)}(a_{i_0} \otimes b_1 a_{i_1} \otimes \dots \otimes b_n a_{i_n}),$$

where $\hat{c} = (c^{(n)}) \in \mathbf{I}^c(A, B)$ is the cumulant function of φ.
We will call $\xi(\nu)$ and $\xi(a_1, \dots, a_m)$ the *cumulants* of ν and $a_1, \dots, a_m$, respectively.

With these notations the free convolution is now described by the following fact.

4.1.7. Theorem. *We have for all $\nu_1, \nu_2 \in \Sigma^{(m)}_B$:*

$$\xi(\nu_1 \boxplus \nu_2) = \xi(\nu_1) + \xi(\nu_2),$$

i.e., for all $n \in \mathbb{N}_0$ and all $i_0, \dots, i_n \in \{1, \dots, m\}$

$$\xi^{(\nu_1 \boxplus \nu_2)}_{n;i_0,\dots,i_n} = \xi^{(\nu_1)}_{n;i_0,\dots,i_n} + \xi^{(\nu_2)}_{n;i_0,\dots,i_n}.$$

Proof. Let, as in the Def. 4.1.4 of the free convolution, $\nu_1 = \nu_{(a_1,\dots,a_m)}$ and $\nu_2 = \nu_{(\tilde{a}_1,\dots,\tilde{a}_m)}$ with $\{a_1, \dots, a_m\}$ and $\{\tilde{a}_1, \dots, \tilde{a}_m\}$ two free sets of random variables in some probability space (A, φ). If we denote by $\hat{c} = (c^{(n)})$ the cumulant function of φ, then, by the definition of freeness, we have for all $n \in \mathbb{N}_0$, all $i_0, \dots, i_n \in \{1, \dots, m\}$, and all $b_1, \dots, b_n \in B$

$$\begin{aligned} \xi^{(\nu_1 \boxplus \nu_2)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) &= c^{(n+1)}\big((a_{i_0} + \tilde{a}_{i_0}) \otimes b_1(a_{i_1} + \tilde{a}_{i_1}) \otimes \dots \otimes b_n(a_{i_n} + \tilde{a}_{i_n})\big) \\ &= c^{(n+1)}(a_{i_0} \otimes b_1 a_{i_1} \otimes \dots \otimes b_n a_{i_n}) \\ &\quad + c^{(n+1)}(\tilde{a}_{i_0} \otimes b_1 \tilde{a}_{i_1} \otimes \dots \otimes b_n \tilde{a}_{i_n}) \\ &= \xi^{(\nu_1)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) + \xi^{(\nu_2)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n). \qquad \square \end{aligned}$$

For $m = 1$, the corresponding quantities ξ_n, given by

$$\xi_n(b_1, \dots, b_n) := c^{(n+1)}(X \otimes b_1 X \otimes b_2 X \otimes \cdots \otimes b_n X),$$

were introduced in [Voi4] in an abstract way (as the main ingredient of the 'canonical form' of a random variable for a given distribution $\nu \in \Sigma_B$) and the foregoing Theorem 4.1.7 reduces in this case to Prop. 3.2 of [Voi4].

4.1.8. NOTATION. For $b_1, \dots, b_m \in B$, we denote by

$$\delta_{(b_1,\dots,b_m)} \in \Sigma_B^{(m)}$$

the special distribution which is determined by the requirements that it is a homomorphism and that

$$\delta_{(b_1,\dots,b_m)}(X_i) = b_i \qquad \text{for } i = 1, \dots, m.$$

Corollary 2.5.5 tells us that we can characterize $\delta_{(b_1,\dots,b_m)}$ also in terms of its cumulants $\xi := \xi(\delta_{(b_1,\dots,b_m)})$ by

$$\xi_{n;i_0,\dots,i_n} \equiv 0 \qquad \text{for } n \neq 0$$

and

$$\xi_{0;i}(1) = \delta_{(b_1,\dots,b_m)}(X_i) = b_i \qquad \text{for } i = 1, \dots, m.$$

Note that the above characterization of $\delta_{(b_1,\dots,b_m)}$ is the same as saying that $\delta_{(b_1,\dots,b_m)}$ is the joint distribution of the m random variables $b_1, \dots, b_m \in (B, \mathrm{id})$, where $\mathrm{id} : B \to B$ is the identity map on B.
It is also clear that $\delta_{(b_1,\dots,b_m)}$ is positive if and only if all b_i are selfadjoint, $b_i^* = b_i$ for all $i = 1, \dots, m$.
If all b_i are zero then we write also

$$\delta_0 := \delta_{(0,\dots,0)} \in \Sigma_B^{(m)+}.$$

4.1.9. PROPOSITION. *1) If $\nu \in \Sigma_B^{(m)}$ is the joint distribution of m random variables $a_1, \dots, a_m \in (A, \varphi)$, then $\nu \boxplus \delta_{(b_1,\dots,b_m)}$ is the joint distribution of the random variables $a_1 + b_1, \dots, a_m + b_m \in (A, \varphi)$,*

$$\nu = \nu_{(a_1,\dots,a_m)} \Longrightarrow \nu \boxplus \delta_{(b_1,\dots,b_m)} = \nu_{(a_1+b_1,\dots,a_m+b_m)}.$$

2) In particular, we have

$$\nu \boxplus \delta_0 = \delta_0 \boxplus \nu = \nu \qquad \textit{for all } \nu \in \Sigma_B^{(m)}.$$

PROOF. 1) This is clear by the Remark 3.3.2 that A and B are free in (A, φ) for each probability space (A, φ) over B.
2) This follows directly by 1). $\square$

The effect of multiplication of a random variable by an element in B, generalizations of Prop. 3.5 and Prop. 3.6 in [Voi4], takes in our frame the following form.

4.1.10. Proposition. *For $m \in \mathbb{N}$, m B-valued random variables $a_1, \dots, a_m \in (A, \varphi)$, and $b \in B$ we have for all $n \in \mathbb{N}_0$, all $i_0, \dots, i_n \in \{1, \dots, m\}$, and all $b_1, \dots, b_n \in B$*

$$\xi^{(ba_1,\dots,ba_m)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) = b\xi^{(a_1,\dots,a_m)}_{n;i_0,\dots,i_n}(b_1 b, \dots, b_n b)$$

and

$$\xi^{(a_1 b,\dots,a_m b)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) = \xi^{(a_1,\dots,a_m)}_{n;i_0,\dots,i_n}(bb_1, \dots, bb_n)b.$$

Proof. We only show the first equality:

$$\begin{aligned}\xi^{(ba_1,\dots,ba_m)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) &= c^{(n+1)}(ba_{i_0} \otimes b_1 ba_{i_1} \otimes \cdots \otimes b_n ba_{i_n}) \\ &= bc^{(n+1)}\big(a_{i_0} \otimes (b_1 b)a_{i_1} \otimes \cdots \otimes (b_n b)a_{i_n}\big) \\ &= b\xi^{(a_1,\dots,a_m)}_{n;i_0,\dots,i_n}(b_1 b, \dots, b_n b). \quad \square\end{aligned}$$

4.1.11. Proposition. *For $m \in \mathbb{N}$, m B-valued random variables $a_1, \dots, a_m \in (A, \varphi)$, and an idempotent $e = e^2 \in B$ we have for all $n \in \mathbb{N}_0$, all $i_0, \dots, i_n \in \{1, \dots, m\}$, and all $b_1, \dots, b_n \in B$*

$$\xi^{(ea_1e,\dots,ea_me)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) = e\xi^{(ea_1e,\dots,ea_me)}_{n;i_0,\dots,i_n}(eb_1e, \dots, eb_ne)e.$$

Proof. We have

$$\begin{aligned}\xi^{(ea_1e,\dots,ea_me)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) &= c^{(n+1)}(ea_{i_0}e \otimes b_1 ea_{i_1}e \otimes \cdots \otimes b_n ea_{i_n}e) \\ &= ec^{(n+1)}\big(ea_{i_0}e \otimes (eb_1e)ea_{i_1}e \otimes \cdots \otimes (eb_ne)ea_{i_n}e\big)e \\ &= e\xi^{(ea_1e,\dots,ea_me)}_{n;i_0,\dots,i_n}(eb_1e, \dots, eb_ne)e. \quad \square\end{aligned}$$

For $m = 1$ and 'symmetric' measures, the description of free convolution can also be put into a relation between formal power series.

4.1.12. Theorem. *For $\nu \in \Sigma_B$ define the formal power series ($b \in B$)*

$$G_\nu(b) := \sum_{n \geq 0} \nu\big(b(Xb)^n\big)$$

and

$$R_\nu(b) := \sum_{n \geq 0} \xi^{(\nu)}_n(\underbrace{b, \dots, b}_{n\text{-times}}).$$

Then we have for all $b \in B$ the relation

$$b\big(1 + R_\nu(G_\nu(b)) \cdot G_\nu(b)\big) = G_\nu(b).$$

Furthermore, for $\nu_1, \nu_2 \in \Sigma_B$ and $b \in B$ we have

$$R_{\nu_1 \boxplus \nu_2}(b) = R_{\nu_1}(b) + R_{\nu_2}(b).$$

One can also rewrite the relation between R_ν and G_ν in the form

$$R_\nu(b) + b^{-1} = K_\nu(b)^{-1},$$

where

$$K_\nu(G_\nu(b)) = G_\nu(K_\nu(b)) = b,$$

if these expressions make sense. In this form, our theorem appeared as Theorem 4.9 in [Voi4].

Proof. We will use Theorem 2.2.3 and define accordingly for $a \in B\langle X\rangle$

$$F(a) := 1 + \sum_{n\geq 1} c_\nu^{(n)}(a^{\otimes n})$$

and

$$G(a) := 1 + \sum_{n\geq 1} \varphi_\nu^{(n)}(a^{\otimes n}) = 1 + \sum_{n\geq 1} \nu(a^n).$$

For $a = Xb$ we get

$$\begin{aligned} F(Xb) &= 1 + \sum_{n\geq 0} c_\nu^{(n+1)}(Xb \otimes Xb \otimes \cdots \otimes Xb) \\ &= 1 + \sum_{n\geq 0} c_\nu^{(n+1)}(X \otimes bX \otimes \cdots \otimes bX)b \\ &= 1 + \sum_{n\geq 0} \xi_n^{(\nu)}(b, \dots, b)b \\ &= 1 + R_\nu(b)b \end{aligned}$$

and

$$\begin{aligned} G_\nu(b) &= \sum_{n\geq 0} \nu\big(b(Xb)^n\big) \\ &= b\sum_{n\geq 0} \nu\big((Xb)^n\big) \\ &= bG(Xb). \end{aligned}$$

Now Theorem 2.2.3 yields for $a = Xb$

$$G(Xb) = F(XbG(Xb)),$$

thus

$$\begin{aligned} G_\nu(b) &= bG(Xb) \\ &= bF(XbG(Xb)) \\ &= bF(XG_\nu(b)) \\ &= b\big(1 + R_\nu(G_\nu(b)) \cdot G_\nu(b)\big). \end{aligned}$$

Note that, at least formally, $G_\nu(b) \in B$.
The statement about $R_{\nu_1 \boxplus \nu_2}$ follows directly from Theorem 4.1.7. $\square$

4.2. B-Gaussian distributions and central limit theorem

One of the fundamental theorems of classical probability theory is the central limit theorem. Here, we will treat the free analogue. As always, one can formulate it either in terms of random variables or in terms of distributions. We choose the second possibility. Thus we have to define the concepts of 'dilation' and 'convergence' of distributions.

4.2.1. Definition. 1) For $\nu \in \Sigma_B^{(m)}$ and $\lambda \in \mathbb{R}$ we define the *dilation* $D_\lambda \nu \in \Sigma_B^{(m)}$ by

$$D_\lambda \nu := \nu_{(\lambda a_1, \dots, \lambda a_m)} \qquad \text{if} \qquad \nu = \nu_{(a_1, \dots, a_m)}$$

(which is, of course, independent of the special choice of $a_1, \dots, a_m$).
2) Let $\nu, \nu_1, \nu_2, \dots \in \Sigma_B^{(m)}$, where B is a Banach algebra. We shall say that ν_k *converges pointwise* to ν, denoted by $\lim_{k\to\infty} \nu_k = \nu$, if we have for each $a \in B\langle X_1, \dots, X_m\rangle$

$$\lim_{k\to\infty} \|\nu_k(a) - \nu(a)\| = 0.$$

4.2.2. Remark. Since cumulants are some polynomials in moments (and vice versa), it is clear that $\lim_{k\to\infty} \nu_k = \nu$ is equivalent to $\lim_{k\to\infty} \xi(\nu_k) = \xi(\nu)$, i.e. to

$$\lim_{k\to\infty} \xi^{(\nu_k)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) = \xi^{(\nu)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n)$$

for all $n \in \mathbb{N}_0$, all $i_0, \dots, i_n \in \{1, \dots, m\}$, and all $b_1, \dots, b_n \in B$.

As in the classical case, quite special distributions will appear in the limit of the central limit theorem.

4.2.3. Definition. Let $m \in \mathbb{N}$ and

$$\eta : B \to M_m(B), \qquad b \mapsto \big(\eta_{ij}(b)\big)_{i,j=1}^m$$

be a linear map. Then we define $\nu_\eta \in \Sigma_B^{(m)}$ by the following requirements on $\xi := \xi(\nu_\eta)$:

$$\xi_{n;i_0,\dots,i_n} \equiv 0 \qquad \text{for } n \neq 1$$

and

$$\xi_{1;i,j}(b) = \nu_\eta(X_i b X_j) = \eta_{ij}(b) \qquad \text{for } i,j \in \{1, \dots, m\} \text{ and } b \in B.$$

The distribution ν_η is called a centered *B-Gaussian distribution* with covariance matrix η.

In [Voi4], a B-Gaussian distribution for $m = 1$ is called B-semicircular.

We will now only treat the most convenient case of a central limit theorem, namely the case of identically distributed random variables. The case $m = 1$ (apart from the positivity statement) is also due to [Voi4].

4.2.4. Theorem. *1) Let B be a Banach algebra and let $\nu \in \Sigma_B^{(m)}$,*

$$\nu : B\langle X_1, \dots, X_m\rangle \to B.$$

If ν is centered, i.e. $\nu(X_i) = 0$ for all $i = 1, \ldots, m$, then

$$\lim_{k\to\infty} D_{1/\sqrt{k}}(\underbrace{\nu \boxplus \ldots \boxplus \nu}_{k\text{-times}}) = \nu_\eta,$$

where $\nu_\eta \in \Sigma_B^{(m)}$ is the centered B-Gaussian distribution with covariance matrix given by

$$\eta_{ij}(b) = \nu(X_i b X_j) \qquad \text{for all } b \in B \text{ and all } i, j = 1, \ldots, m.$$

2) Let B be a unital C^*-algebra. If ν is positive, then the corresponding central limit distribution ν_η is positive, too,

$$\nu \in \Sigma_B^{(m)+} \Longrightarrow \nu_\eta \in \Sigma_B^{(m)+}.$$

Proof. 1) If we denote

$$\xi^{(k)} := \xi(D_{1/\sqrt{k}}(\nu \boxplus \ldots \boxplus \nu)), \qquad \text{in particular } \xi := \xi^{(1)} = \xi(\nu)$$

and

$$\xi^{(\eta)} := \xi(\nu_\eta),$$

then the assertion is equivalent to

$$\lim_{k\to\infty} \xi^{(k)}_{n;i_0,\ldots,i_n}(b_1, \ldots, b_n) = \xi^{(\eta)}_{n;i_0,\ldots,i_n}(b_1, \ldots, b_n)$$

for all $\in \mathbb{N}_0$, all $i_0, \ldots, i_n \in \{1, \ldots, m\}$, and all $b_1, \ldots, b_n \in B$.
Since in general

$$\xi^{(\nu_1 \boxplus \nu_2)}_{n;i_0,\ldots,i_n} = \xi^{(\nu_1)}_{n;i_0,\ldots,i_n} + \xi^{(\nu_2)}_{n;i_0,\ldots,i_n}$$

and

$$\xi^{(D_\lambda \nu)}_{n;i_0,\ldots,i_n} = \lambda^{n+1} \xi^{(\nu)}_{n;i_0,\ldots,i_n},$$

we get

$$\xi^{(k)}_{n;i_0,\ldots,i_n} = \frac{1}{k^{(n+1)/2}} k \xi_{n;i_0,\ldots,i_n} = \frac{1}{k^{(n-1)/2}} \xi_{n;i_0,\ldots,i_n},$$

and thus

$$\lim_{k\to\infty} \xi^{(k)}_{n;i_0,\ldots,i_n}(b_1, \ldots, b_n) = 0$$

for $n \geq 2$ and all $b_1, \ldots, b_n \in B$. Since ν is centered, we also have

$$\xi^{(k)}_{0;i}(1) = \sqrt{k}\xi_{0;i}(1) = \sqrt{k}\nu(X_i) = 0$$

for all $i = 1, \ldots, m$ and all $k \in \mathbb{N}$. Thus only the case $n = 1$ gives a non-vanishing contribution. In that case

$$\lim_{k\to\infty} \xi_{1;i,j}(b) = \xi_{1;i,j}(b) = \nu(X_i b X_j) = \eta_{ij}(b).$$

2) By Corollary 4.1.5, the free convolution preserves positivity. Since dilation and pointwise limit have the same property, the assertion is clear. □

4.3. Positivity of B-Gaussian distributions

¿From the probabilistic point of view, only positive distributions are of interest, so we should check under what conditions on η the B-Gaussian distribution ν_η is positive. By part 2) of our central limit theorem 4.2.4, this is exactly the case if there exists a (centered) positive distribution $\nu \in \Sigma_B^{(m)+}$ which has η as covariance matrix, i.e. with

$$\nu(X_i b X_j) = \eta_{ij}(b) \qquad \text{for all } b \in B \text{ and all } i,j = 1,\dots,m.$$

In this section, we assume always that B is a unital C^*-algebra.

4.3.1. THEOREM. *The B-Gaussian distribution $\nu_\eta \in \Sigma_B^{(m)}$ is positive if and only if its covariance matrix $\eta : B \to M_m(B)$ has the following property: For each $n \in \mathbb{N}$, all $b_1,\dots,b_n \in B$, and all $r(1),\dots,r(n) \in \{1,\dots,m\}$ the matrix*

$$\big(\eta_{r(i)r(j)}(b_i b_j^*)\big)_{i,j=1}^n \in M_n(B)$$

is positive.

PROOF. First, we prove the necessity of the stated condition. Let $\nu_\eta \in \Sigma_B^{(m)+}$ be positive. Then, for $n \in \mathbb{N}$, $b_1,\dots,b_n \in B$, and all $r(1),\dots,r(n) \in \{1,\dots,m\}$ the matrix

$$\big(\eta_{r(i)r(j)}(b_i b_j^*)\big)_{i,j=1}^n = \big(\nu_\eta(X_{r(i)} b_i b_j^* X_{r(j)})\big)_{i,j=1}^n \in M_n(B)$$

is positive by Lemma 3.5.2 (with $a_i := X_{r(i)} b_i$).
Now we want to see that the condition is also sufficient. Let $\eta : B \to M_m(B)$ be given which fulfills the condition of the theorem. By part 2) of our central limit theorem 4.2.4, we are done if we can present a positive distribution $\nu \in \Sigma_B^{(m)+}$ with the given η as covariance matrix. We will construct such a distribution as the distribution of the sum of 'creation' and 'annihilation' operators on a 'degenerate' Fock space.
The degenerate Fock space $\mathcal{F}$ is the B-B-bimodule

$$\mathcal{F} := B \oplus B\mathcal{X}_m B,$$

where

$$\mathcal{X}_m = \operatorname{span}\{X_1,\dots,X_m\}$$

and

$$B\mathcal{X}_m B := \{\sum_{i=1}^n b_i X_{r(i)} \tilde{b}_i \mid b_i, \tilde{b}_i \in B,\ n \in \mathbb{N}, r(1),\dots,r(n) \in \{1,\dots,m\}\},$$

equipped with a B-valued inner product

$$\langle\ ,\ \rangle : \mathcal{F} \times \mathcal{F} \to B$$

given by linear extension of

$$\langle b_0 + b_1 X_i b_2, \tilde{b}_0 + \tilde{b}_1 X_j \tilde{b}_2 \rangle := b_0^* \tilde{b}_0 + b_2^* \eta_{ij}(b_1^* \tilde{b}_1) \tilde{b}_2.$$

Let us, for each $i = 1, \dots, m$, define a creation operator l_i^* and an annihilation operator l_i on $\mathcal{F}$ by ($b, b_1, b_2 \in B$, $j = 1, \dots, m$)

$$\begin{aligned} l_i^* b &= X_i b = 1X_i b \\ l_i^* b_1 X_j b_2 &= 0 \end{aligned}$$

and

$$\begin{aligned} l_i b &= 0 \\ l_i b_1 X_j b_2 &= \eta_{ij}(b_1) b_2. \end{aligned}$$

The operators l_i and l_i^* are adjoints with respect to $\langle\, ,\, \rangle$, i.e.

$$\langle l_i f_1, f_2 \rangle = \langle f_1, l_i^* f_2 \rangle \qquad \text{for all } f_1, f_2 \in \mathcal{F}.$$

Furthermore, we identify elements of B with the corresponding multiplication operators from the left on $\mathcal{F}$. Note that then b and b^* are adjoints as multiplication operators.

We take now as probability space the pair (A, φ), where A is the $*$-algebra generated by all creation and annihilation operators l_i^* and l_i, respectively, for $i = 1, \dots, m$ and by all multiplication operators from B, and where φ is the B-functional

$$\begin{aligned} \varphi : A &\to B \\ a &\mapsto \langle 1, a1 \rangle. \end{aligned}$$

Note that

$$\langle f_1 b_1, f_2 b_2 \rangle = b_1^* \langle f_1, f_2 \rangle b_2 \qquad \text{for all } f_1, f_2 \in \mathcal{F}$$

implies

$$\varphi(b_1 a b_2) = \langle 1, b_1 a b_2 1 \rangle = b_1 \langle 1, a1 \rangle b_2 = b_1 \varphi(a) b_2$$

for all $a \in A$ and all $b_1, b_2 \in B$.

As the wanted ν we take now the distribution of the random variables $l_1 + l_1^*, \dots, l_m + l_m^* \in (A, \varphi)$, i.e.

$$\nu := \nu_{(l_1 + l_1^*, \dots, l_m + l_m^*)}.$$

This distribution has the right covariance matrix because ($b \in B$, $i, j = 1, \dots, m$)

$$\begin{aligned} \nu(X_i b X_j) &= \varphi\big((l_i + l_i^*) b (l_j + l_j^*)\big) \\ &= \langle 1, (l_i + l_i^*) b (l_j + l_j^*) 1 \rangle \\ &= \langle X_i, b X_j \rangle \\ &= \eta_{ij}(b). \end{aligned}$$

So it remains to show that φ is positive. This follows from our condition on η as can be seen as follows. Consider $a \in A$. We have to show that $\varphi(aa^*) \in B$ is positive. We can write $a^*1 \in \mathcal{F}$ in the form

$$a^* 1 = b + \sum_{i=1}^{n} b_i X_{r(i)} \tilde{b}_i \qquad \text{with } b, b_i, \tilde{b}_i \in B,\ r(1), \dots, r(n) \in \{1, \dots, m\}$$

and obtain

$$\begin{aligned}\varphi(aa^*) &= \langle 1, aa^*1\rangle \\ &= \langle a^*1, a^*1\rangle \\ &= \langle b + \sum_{i=1}^n b_i X_{r(i)} \tilde{b}_i, b + \sum_{j=1}^n b_j X_{r(j)} \tilde{b}_j\rangle \\ &= b^*b + \sum_{i,j=1}^n \tilde{b}_i^* \eta_{r(i)r(j)}(b_i^* b_j)\tilde{b}_j.\end{aligned}$$

By our assumption on η, the matrix

$$\big(\eta_{r(i)r(j)}(b_i^* b_j)\big)_{i,j=1}^n \in M_n(B)$$

is positive and its entries can thus, by Lemma 3.5.2, be written as

$$\eta_{r(i)r(j)}(b_i^* b_j) = \sum_{k=1}^n b_i^{(k)*} b_j^{(k)}$$

for some $b_i^{(k)} \in B$ $(i, k = 1, \dots, n)$. Hence

$$\begin{aligned}\varphi(aa^*) &= b^*b + \sum_{i,j=1}^n \sum_{k=1}^n \tilde{b}_i^* b_i^{(k)*} b_j^{(k)} \tilde{b}_j \\ &= b^*b + \sum_{k=1}^n \big(\sum_{i=1}^n b_i^{(k)}\tilde{b}_i\big)^* \big(\sum_{j=1}^n b_j^{(k)}\tilde{b}_j\big) \\ &\geq 0. \quad \square\end{aligned}$$

4.3.2. Remarks. 1) Instead of the degenerate Fock space we could also consider the full Fock space. There, the vacuum expectation yields directly the B-Gaussian distribution. Here we only give the relevant definitions and facts without proof. A more extensive analysis of the full Fock space will follow in section 4.6.
The full Fock space is nothing else but

$$B\langle \mathcal{X}_m\rangle = B\langle X_1, \dots, X_m\rangle = B \oplus B\mathcal{X}_m B \oplus B\mathcal{X}_m B\mathcal{X}_m B \oplus \dots$$

equipped with the B-valued inner product

$$\langle\ ,\ \rangle : B\langle \mathcal{X}_m\rangle \times B\langle \mathcal{X}_m\rangle \to B$$

given by linear extension of

$$\big\langle b_0 X_{i_0} b_1 X_{i_1} \dots b_n X_{i_n} b_{n+1}, \tilde{b}_0 X_{j_0} \tilde{b}_1 X_{j_1} \dots \tilde{b}_k X_{j_k} \tilde{b}_{k+1}\big\rangle := \\ \delta_{nk} b_{n+1}^* \eta_{i_n j_n}\Big(b_n^* \dots \eta_{i_1 j_1}\big(b_1^* \eta_{i_0 j_0}(b_0^* \tilde{b}_0)\tilde{b}_1\big)\dots \tilde{b}_n\Big)\tilde{b}_{n+1}.$$

Again, we have creation operators l_i^*,

$$l_i^* b_0 X_{i_0} \dots X_{i_n} b_{n+1} := X_i b_0 X_{i_0} \dots X_{i_n} b_{n+1},$$

and annihilation operators l_i,

$$l_i b = 0$$
$$l_i b_0 X_{i_0} b_1 X_{i_1} \dots X_{i_n} b_{n+1} = \big(\eta_{i i_0}(b_0) b_1\big) X_{i_1} \dots X_{i_n} b_{n+1},$$

which are adjoints with respect to $\langle\, , \rangle$.
Let A be the $*$-algebra generated by B (acting as multiplication operators) and by all creation and annihilation operators. Then the vacuum expectation

$$\varphi : A \to B$$
$$a \mapsto \langle 1, a1 \rangle$$

is positive if η fulfills the condition of Theorem 4.3.1. The distribution of the random variables $l_1 + l_1^*, \dots, l_m + l_m^* \in (A, \varphi)$ gives the B-Gaussian distribution ν_η, i.e.

$$\nu_{(l_1+l_1^*,\dots,l_m+l_m^*)} = \nu_\eta.$$

2) Let us denote by p_i the operator

$$p_i := l_i + l_i^* + l_i^* l_i \qquad (i = 1, \dots, m)$$

on the full Fock space $B\langle \mathcal{X}_m \rangle$. Then we could, in analogy to the scalar situation on the symmetric Fock space (cf. [HP,Par,Mey,Küm2,FM]), call the joint distribution π_η of $p_1, \dots, p_m \in (A, \varphi)$ a centered *Poisson distribution* (with 'jump size' η). It is characterized by

$$\xi_{0;i} \equiv 0 \qquad \text{for all } i = 1, \dots, m$$
$$\xi_{n;i_0,\dots,i_n}(b_1, \dots, b_n) = \eta_{i_0 i_1}(b_1) \eta_{i_1 i_2}(b_2) \dots \eta_{i_{n-1} i_n}(b_n),$$

where $\xi := \xi(\pi_\eta)$. A more general class of Poisson distributions will be considered in the next section, and in section 4.7 it will be shown that they have essentially a structure like in the above example.

4.4. Compound B-Poisson distributions

A very important method in classical probability theory for generating distributions centers around a generalization of the Poisson distribution. Given a random variable Y, one can construct out of this a so called compound Poisson distribution π_Y by the prescription that the moments of Y give, up to a common factor, the cumulants of π_Y. One possibility to make the transition from Y to π_Y is by a limit theorem, which has as an imediate consequence that positivity of the distribution of Y implies positivity of π_Y. The importance of these compound Poisson distributions rises from the fact that all infinitely divisible distributions can be approximated by compound Poisson distributions (and more generally, all processes with independent and stationary increments can be approximated by compound Poisson processes), see [Fel].

These ideas can be imitated in our case.

4.4.1. Definition. For a distribution $\nu \in \Sigma_B^{(m)}$ and a real number $\lambda \in \mathbb{R}$ we define the *compound B-Poisson distribution* $\pi_{\nu,\lambda} \in \Sigma_B^{(m)}$ by (with $\xi := \xi(\pi_{\nu,\lambda})$)

$$\xi_{n;i_0,\dots,i_n}(b_1, \dots, b_n) := \lambda \nu(X_{i_0} b_1 X_{i_1} \dots b_n X_{i_n})$$

for each $n \in \mathbb{N}_0$, all $i_0, \dots, i_n \in \{1, \dots, m\}$, and all $b_1, \dots, b_n \in B$.

4.4.2. Notations. 1) Let us denote by $\delta_0 \in \Sigma_B^{(m)+}$ the special distribution given by $\xi(\delta_0) \equiv 0$, i.e.

$$\delta_0(b) = b \qquad \text{for all } b \in B$$

and

$$\delta_0(X_{i_0} b_1 \dots b_n X_{i_n}) = 0$$

for each $n \in \mathbb{N}_0$, all $i_0, \dots, i_n \in \{1, \dots, m\}$, and all $b_1, \dots, b_n \in B$ (compare Notation 4.1.8).
2) For $\lambda \in \mathbb{R}$, the combination $\lambda\nu_1 + (1-\lambda)\nu_2 \in \Sigma_B^{(m)}$ of two distributions $\nu_1, \nu_2 \in \Sigma_B^{(m)}$ is of course defined by

$$(\lambda\nu_1 + (1-\lambda)\nu_2)(a) = \lambda\nu_1(a) + (1-\lambda)\nu_2(a) \qquad \text{for all } a \in B\langle X_1, \dots, X_m\rangle.$$

We have of course

$$\nu_1, \nu_2 \in \Sigma_B^{(m)+}, 0 \le \lambda \le 1 \Longrightarrow \lambda\nu_1 + (1-\lambda)\nu_2 \in \Sigma_B^{(m)+}.$$

Now we can formulate our limit theorem for the compound B-Poisson distribution.

4.4.3. Theorem. *Let a distribution $\nu \in \Sigma_B^{(m)}$ and a parameter $\lambda \in \mathbb{R}$ be given. With the definition*

$$\nu_k := (1 - \frac{\lambda}{k})\delta_0 + \frac{\lambda}{k}\nu$$

we have

$$\lim_{k\to\infty} \underbrace{\nu_k \boxplus \dots \boxplus \nu_k}_{k\text{-times}} = \pi_{\nu,\lambda}.$$

Proof. We have for $n \in \mathbb{N}_0$, $i_0, \dots, i_n \in \{1, \dots, m\}$, and $b_1, \dots, b_n \in B$

$$\begin{aligned}\xi^{(\nu_k)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) &= c^{(n+1)}_{\nu_k}(X_{i_0} \otimes b_1X_{i_1} \otimes \dots \otimes b_nX_{i_n}) \\ &= \frac{\lambda}{k}\nu(X_{i_0}b_1X_{i_1} \dots b_nX_{i_n}) + O(1/k^2),\end{aligned}$$

thus

$$\begin{aligned}\xi^{(\nu_k \boxplus \dots \boxplus \nu_k)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) &= k\xi^{(\nu_k)}_{n;i_0,\dots,i_n}(b_1, \dots, b_n) \\ &= \lambda\nu(X_{i_0}b_1X_{i_1} \dots b_nX_{i_n}) + O(1/k),\end{aligned}$$

from which the assertion follows. □

Since for $\lambda \in \mathbb{R}_+$ and $k > \lambda$ positivity is preserved in the course of this limit theorem, we get the following corollary.

4.4.4. Corollary. *Let B be a C^*-algebra. If $\nu \in \Sigma_B^{(m)}$ is positive and $\lambda \ge 0$, then $\pi_{\nu,\lambda} \in \Sigma_B^{(m)}$ is also positive,*

$$\nu \in \Sigma_B^{(m)+}, \lambda \ge 0 \Longrightarrow \pi_{\nu,\lambda} \in \Sigma_B^{(m)+}.$$

4.5. Infinitely divisible distributions

We will now introduce the analogue of infinitely divisible distributions. Our aim is to see that they can be described as in the classical case, namely we shall show that infinitely divisible distributions can be approximated by compound B-Poisson distributions and that one can realize each infinitely divisible distribution by a special random variable on a full Fock space.

The notion of infinite divisibility has only a non-trivial meaning if we stay within the set of positive distributions. Thus, in this context we will always require B to be a C^*-algebra.

We will denote our infinitely divisible distributions by μ, which should not be confused with the Möbius function.

4.5.1. Definition. Let B be a C^*-algebra. Then a positive distribution $\mu \in \Sigma_B^{(m)+}$ is called *infinitely divisible*, if for each $k \in \mathbb{N}$ there exists a positive distribution $\mu_{1/k} \in \Sigma_B^{(m)+}$ such that

$$\mu = \underbrace{\mu_{1/k} \boxplus \ldots \boxplus \mu_{1/k}}_{k\text{-times}}.$$

4.5.2. Remark. Note that on the level of the cumulants the connection between μ and $\mu_{1/k}$ is given by $\xi(\mu) = k\xi(\mu_{1/k})$. Thus $\xi(\mu_{1/k}) = (1/k)\xi(\mu)$, and hence also $\mu_{1/k}$ itself, is uniquely determined. The crucial requirement is positivity.

4.5.3. Proposition. *Let $\mu \in \Sigma_B^{(m)+}$ be infinitely divisible. Then there exists a family $\{\mu_t \mid t \geq 0\} \subset \Sigma_B^{(m)+}$ of positive distributions with*

i) $\mu_0 = \delta_0$, $\mu_1 = \mu$
ii) $\mu_{t+s} = \mu_t \boxplus \mu_s$ *for* $s, t \geq 0$
iii) *the map* $t \mapsto \mu_t$ *is pointwise continuous.*

Proof. We define μ_t by $\xi(\mu_t) := t\xi(\mu)$. The requirements i), ii), and iii) are then fulfilled, one only has to think about positivity of the μ_t. By definition of 'infinitely divisible', μ_t is positive for all t of the form $t = 1/k$ ($k \in \mathbb{N}$) and hence, because of

$$\mu_{p/q} = \underbrace{\mu_{1/q} \boxplus \ldots \boxplus \mu_{1/q}}_{p\text{-times}},$$

also for all rational t. The positivity for all $t \geq 0$ follows then by continuity. □

4.5.4. Proposition. *Let $\mu \in \Sigma_B^{(m)+}$ be infinitely divisible with corresponding semi-group $\{\mu_t \mid t \geq 0\}$. Then the cumulants of μ are determined by*

$$\xi(\mu) = \lim_{t \to 0} \frac{1}{t} \mu_t,$$

i.e. for all $n \in \mathbb{N}_0$, $i_0, \ldots, i_n \in \{1, \ldots, m\}$, and $b_1, \ldots, b_n \in B$ we have

$$\xi^{(\mu)}_{n;i_0,\ldots,i_n}(b_1, \ldots, b_n) = \lim_{t \to 0} \frac{1}{t} \mu_t(X_{i_0} b_1 X_{i_1} \ldots b_n X_{i_n}).$$

Proof. Since

$$c^{(n+1)}_{\mu_t} = t c^{(n+1)}_{\mu} \qquad \text{for all } n \in \mathbb{N}_0,$$

we have for $n \in \mathbb{N}_0$, $i_0, \dots, i_n \in \{1, \dots, m\}$, and $b_1, \dots, b_n \in B$

$$\begin{aligned}\frac{1}{t}\mu_t(X_{i_0}b_1X_{i_1}\dots b_nX_{i_n}) &= \frac{1}{t}[tc_\mu^{(n+1)}(X_{i_0}\otimes b_1X_{i_1}\otimes\dots\otimes b_nX_{i_n}) + O(t^2)]\\ &= c_\mu^{(n+1)}(X_{i_0}\otimes b_1X_{i_1}\otimes\dots\otimes b_nX_{i_n}) + O(t)\\ &= \xi^{(\mu)}_{n;i_0,\dots,i_n}(b_1,\dots,b_n) + O(t),\end{aligned}$$

which implies the assertion □

4.5.5. Theorem. *The infinitely divisible distributions $\mu \in \Sigma_B^{(m)+}$ are exactly those distributions which are pointwise limits of compound B-Poisson distributions $\pi_{\nu,\lambda}$ with $\nu \in \Sigma_B^{(m)+}$ and $\lambda \geq 0$.*

Proof. Let $\nu \in \Sigma_B^{(m)+}$ and $\lambda \geq 0$. Then the compound B-Poisson distribution $\pi_{\nu,\lambda} \in \Sigma_B^{(m)+}$ is infinitely divisible, because for all $k \in \mathbb{N}$

$$\pi_{\nu,\lambda} = \underbrace{\pi_{\nu,\lambda/k} \boxplus \dots \boxplus \pi_{\nu,\lambda/k}}_{k\text{-times}} \qquad \text{and} \qquad \pi_{\nu,\lambda/k} \in \Sigma_B^{(m)+}.$$

Furthermore, infinite divisibility is preserved under pointwise limits.
So it remains to show that each infinitely divisible $\mu \in \Sigma_B^{(m)+}$ can be written in the form

$$\mu = \lim_{k\to\infty} \pi_{\nu_k,\lambda_k} \qquad \text{with } \nu_k \in \Sigma_B^{(m)+} \text{ and } \lambda_k \geq 0.$$

Let $\mu \in \Sigma_B^{(m)+}$ be infinitely divisible with the corresponding semi-group $\{\mu_t \mid t \geq 0\} \subset \Sigma_B^{(m)+}$. Then we put

$$\nu_k := \mu_{1/k} \in \Sigma_B^{(m)+} \qquad \text{and} \qquad \lambda_k := k \geq 0.$$

Showing that $\mu = \lim_{k\to\infty} \pi_{\mu_{1/k},k}$ is the same as proving $\xi(\mu) = \lim_{k\to\infty} \xi(\pi_{\mu_{1/k},k})$. But since $\xi(\pi_{\mu_t,1/t})$ is, just by definition, nothing else than $\frac{1}{t}\mu_t$, the assertion follows by the foregoing Prop. 4.5.4. □

4.6. Full Fock space over a Hilbert-B-bimodule

Before being able to fulfill our promise on the special realizations of infinitely divisible distributions we have to make some general investigations on the full Fock space over a Hilbert-B-bimodule and the corresponding creation, annihilation, and preservation operators.

We define the notion of a Hilbert-B-bimodule in a form which is just adapted to our purposes. Similar notations have appeared in [Pas,Rie] or in the context of KK-theory for operator algebras [Kas,Bla].

The construction of a full Fock space and the corresponding creation, annihilation and preservation operators is modelled according to the scalar-valued theory, see [Eva,Voi1,Spe1]. For the bosonic scalar analogue of the preservation operator one should compare [HP,Par,Mey].

Voiculescu introduced in [Voi4] also a similar kind of full Fock space, but, since he does not consider positivity questions in this context, without specifying an inner product. Recently, Pimsner [Pim] introduced creation and annihilation operators on a full Fock space and considered the corresponding C^*-algebras. Some kind of free noise on a Hilbert module appeared also in [AL].

In the following, B will be a fixed C^*-algebra.

4.6.1. Definition. Let M be a B-B-bimodule. A sesquilinear form (linearity with respect to $\mathbb{C}$!)

$$\langle\ ,\ \rangle : M \times M \to B$$

is called *B-valued inner product*, if we have for all $m_1, m_2 \in M$ and all $b, b_1, b_2 \in B$

$$\langle bm_1b_1, m_2b_2\rangle = b_1^*\langle m_1, b^*m_2\rangle b_2$$

and

$$\langle m_1, m_2\rangle^* = \langle m_2, m_1\rangle,$$

and if

$$\langle m, m\rangle \geq 0 \qquad \text{for all } m \in M.$$

In this situation, we call the pair $(M,\text{Ł}\,,\ \rangle)$ a *Hilbert-B-bimodule.* If we have

$$\langle m, m\rangle \neq 0 \qquad \text{for all } m \in M \text{ with } m \neq 0,$$

then we call the Hilbert-B-bimodule $(M,\text{Ł}\,,\ \rangle)$ *non-degenerate.*

4.6.2. Remarks. 1) As for conditional expectations, we have that positivity implies complete positivity, i.e. we have for a Hilbert-B-bimodule $(M,\text{Ł}\,,\ \rangle)$ that for each $n \in \mathbb{N}$ and all $m_1, \dots, m_n \in M$ the matrix

$$\big(\langle m_i, m_j\rangle\big)_{i,j=1}^n \in M_n(B)$$

is positive. This follows directly by Lemma 3.5.3.

2) Our definition includes also the following implication: If we have for a $m_1 \in M$ that $\langle m_1, m_1\rangle = 0$ then we have $\langle m_1, m_2\rangle = 0$ for all $m_2 \in M$. This can be seen as follows: By 1), the matrix $\big(\langle m_i, m_j\rangle\big)_{i,j=1,2} \in M_2(B)$ is positive, so we can write by Lemma 3.5.2

$$\langle m_i, m_j\rangle = b_i^{(1)}b_j^{(1)*} + b_i^{(2)}b_j^{(2)*} \qquad \text{for all } i, j = 1, 2$$

with some $b_i^{(k)} \in B$ $(i, k = 1, 2)$. But

$$0 = \langle m_1, m_1\rangle = b_1^{(1)}b_1^{(1)*} + b_1^{(2)}b_1^{(2)*}$$

implies $b_1^{(1)} = b_1^{(2)} = 0$, and hence

$$\langle m_1, m_2\rangle = b_1^{(1)}b_2^{(1)*} + b_1^{(2)}b_2^{(2)*} = 0.$$

3) If $(M_0, \langle\ ,\ \rangle_0)$ is a Hilbert-B-bimodule, then the last remark shows that we can divide out the null space in order to get a non-degenerate Hilbert-B-bimodule: Define

$$N := \{n \in M_0 \mid \langle n, n\rangle_0 = 0\} \subset M_0.$$

By 2), we have $\langle N, M_0\rangle_0 = 0$, which shows that $N + N = N$ and, because of

$$\langle b_1nb_2, b_1nb_2\rangle_0 = b_2^*\langle n, b_1^*b_1nb_2\rangle_0 = 0$$

for $n \in N$ and $b_1, b_2 \in B$, that $BNB = N$. Thus N is a submodule of M_0 and we can consider

$$M := M_0/N = \{m + N \mid m \in M_0\}$$

equipped with the B-valued inner product

$$\langle m_1 + N, m_2 + N\rangle := \langle m_1, m_2\rangle_0.$$

Then $(M, \langle\, ,\, \rangle)$ is a non-degenerate Hilbert-B-bimodule. We will need this construction in the next section.

4) Note that our notion of a Hilbert-B-bimodule is not quite standard. Since we work totally on the algebraic level we do not bother about whether M is closed or not. Thus a more correct name for our object would be 'preHilbert-B-bimodule'. Usually, also non-degenerateness is included in the definition of a preHilbert-bimodule and left and right action are not dealt with in a symmetric way. Instead one starts from a non-degenerate right Hilbert module structure, defines 'bounded operators' and can then consider a left module action given by some homomorphism ψ from B to the bounded operators on the module. For more information on Hilbert modules and bimodules we refer to [Pas,Rie,Kas,Bla].

4.6.3. Notation. By $\mathcal{L}(M, M)$ we denote the $\mathbb{C}$-linear maps from M into itself. The algebra B will be considered as a subset of $\mathcal{L}(M, M)$ by identifying an element $b \in B$ with the operator $\lambda(b)$ acting by left multiplication with b. Note that then b and b^* are adjoints as operators in $\mathcal{L}(M, M)$, namely

$$\langle\lambda(b)m_1, m_2\rangle = \langle bm_1, m_2\rangle = \langle m_1, b^*m_2\rangle = \langle m_1, \lambda(b^*)m_2\rangle$$

for all $m_1, m_2 \in M$.

4.6.4. Proposition. *Let $(M, \langle\, ,\, \rangle)$ be a Hilbert-B-bimodule and let $m \in M$ be a distinguished element with $\langle m, m\rangle = 1 \in B$ and $bm = mb$ for all $b \in B$. Let $A \subset \mathcal{L}(M, M)$ be a $*$-algebra containing the multiplication operators from B, $1 \in B \subset A$, such that*

$$\langle am_1, m_2\rangle = \langle m_1, a^*m_2\rangle \qquad \text{for all } a \in A \text{ and all } m_1, m_2 \in M.$$

Define

$$\varphi : A \to B$$
$$a \mapsto \langle m, am\rangle.$$

Then φ is a positive B-functional.

Proof. We have for $a \in A$ and $b_1, b_2 \in B$

$$\begin{aligned}\varphi(b_1ab_2) &= \langle m, b_1ab_2m\rangle \\ &= \langle b_1^*m, ab_2m\rangle \\ &= \langle mb_1^*, a(mb_2)\rangle \\ &= b_1\langle m, a(mb_2)\rangle \\ &= b_1\langle a^*m, mb_2\rangle \\ &= b_1\langle a^*m, m\rangle b_2 \\ &= b_1\varphi(a)b_2,\end{aligned}$$

and for $b \in B$

$$\varphi(b) = \langle m, bm \rangle = \langle m, mb \rangle = \langle m, m \rangle b = b,$$

hence φ is a B-functional.
Consider now $a \in A$ and put $\tilde{m} := am \in M$. Then

$$\varphi(aa^*) = \langle a^* m, a^* m \rangle = \langle \tilde{m}, \tilde{m} \rangle \geq 0$$

and thus φ is positive. □

4.6.5. Definition. Given a Hilbert-B-bimodule $(M, \text{Ł}\,, \rangle)$, we define the *full Fock space over M* as the B-B-bimodule

$$\mathcal{F}(M) := \bigoplus_{k=0}^{\infty} M^{\otimes_B k} = B \oplus M \oplus (M \otimes_B M) \oplus (M \otimes_B M \otimes_B M) \oplus \dots$$

equipped with

$$\langle \ , \ \rangle : \mathcal{F}(M) \times \mathcal{F}(M) \to B$$

given by linear extension of ($b, \tilde{b} \in B$, $m_i, \tilde{m}_j \in M$)

$$\langle b, \tilde{b} \rangle := b^* \tilde{b}$$

and

$$\begin{aligned}
&\langle m_1 \otimes \dots \otimes m_k, \tilde{m}_1 \otimes \dots \otimes \tilde{m}_l \rangle := \\
&\qquad = \delta_{kl} \langle m_k, \langle m_1 \otimes \dots \otimes m_{k-1}, \tilde{m}_1 \otimes \dots \otimes \tilde{m}_{k-1} \rangle \tilde{m}_k \rangle \\
&\qquad = \delta_{kl} \Big\langle m_k, \langle m_{k-1}, \dots, \langle m_1, \tilde{m}_1 \rangle \dots \tilde{m}_{k-1} \rangle \tilde{m}_k \Big\rangle.
\end{aligned}$$

4.6.6. Proposition. *If $(M, \text{Ł}\,, \rangle)$ is a Hilbert-B-bimodule, then $(\mathcal{F}(M), \langle\, , \rangle)$ is a Hilbert-B-bimodule, too.*

Proof. Only the positivity of our inner product needs some reflection. It suffices to show, for fixed $m_{l,r} \in M$, the positivity of

$$\langle \sum_r m_{1,r} \otimes \dots \otimes m_{k,r}, \sum_s m_{1,s} \otimes \dots \otimes m_{k,s} \rangle \in B.$$

This can be shown in the same way as in the proof of Theorem 3.5.6 (cf. also

Theorem 4.3.1), namely, for some $b_r^{(p_1,\dots,p_k)} \in B$, we have

$$
\begin{aligned}
\sum_{r,s} \langle m_{1,r} \otimes \dots \otimes m_{k,r}, m_{1,s} \otimes \dots \otimes m_{k,s} \rangle = & \\
&= \sum_{r,s} \Big\langle m_{k,r}, \dots, \big\langle m_{2,r}, \langle m_{1,r}, m_{1,s} \rangle m_{2,s} \big\rangle \dots m_{k,s} \Big\rangle \\
&= \sum_{r,s} \langle m_{k,r}, \dots, \langle m_{2,r}, \sum_{p_1} b_r^{(p_1)*} b_s^{(p_1)} m_{2,s} \rangle \dots m_{k,s} \rangle \\
&= \sum_{r,s} \sum_{p_1} \langle m_{k,r}, \dots, \langle b_r^{(p_1)} m_{2,r}, b_s^{(p_1)} m_{2,s} \rangle \dots m_{k,s} \rangle \\
&= \sum_{r,s} \sum_{p_1} \langle m_{k,r}, \dots, \sum_{p_2} b_r^{(p_1,p_2)*} b_s^{(p_1,p_2)} \dots m_{k,s} \rangle \\
&= \dots \\
&= \sum_{r,s} \sum_{p_1,\dots,p_k} b_r^{(p_1,\dots,p_k)*} b_s^{(p_1,\dots,p_k)} \\
&= \sum_{p_1,\dots,p_k} \big(\sum_r b_r^{(p_1,\dots,p_k)} \big)^* \big(\sum_s b_s^{(p_1,\dots,p_k)} \big) \\
&\geq 0. \quad \square
\end{aligned}
$$

4.6.7. Remark. Note that non-degenerateness of $(M, \text{Ł}\,, \rangle)$ does not necessarily imply non-degenerateness of $(\mathcal{F}(M), \langle\,,\,\rangle)$. If, for example, $\langle m_1, m_1 \rangle = b^* b \neq 0$ $(b \in B)$ and $b m_2 = 0$, then

$$\langle m_1 \otimes m_2, m_1 \otimes m_2 \rangle = \big\langle m_2, \langle m_1, m_1 \rangle m_2 \big\rangle = \langle m_2, b^* b m_2 \rangle = 0,$$

but in general there is no reason why $m_1 \otimes m_2$ should vanish.

4.6.8. Definition. Let $(M, \text{Ł}\,, \rangle)$ be a Hilbert-B-bimodule. We define, for each $m \in M$, a *creation operator* $l^*(m)$ and an *annihilation operator* $l(m)$ on $\mathcal{F}(M)$. They are given by $(b \in B,\ m_1, \dots, m_n \in M)$

$$
\begin{aligned}
l^*(m) b &= mb \\
l^*(m) m_1 \otimes \dots \otimes m_n &= m \otimes m_1 \otimes \dots \otimes m_n
\end{aligned}
$$

and

$$
\begin{aligned}
l(m) b &= 0 \\
l(m) m_1 \otimes m_2 \otimes \dots \otimes m_n &= \langle m, m_1 \rangle m_2 \otimes \dots \otimes m_n .
\end{aligned}
$$

As usual, B will be identified with its action on $\mathcal{F}(M)$ by left multiplication. Then one has $(b_1, b_2 \in B,\ m \in M)$

$$l^*(b_1 m b_2) = b_1 l^*(m) b_2 \qquad \text{and} \qquad l(b_1 m b_2) = b_2^* l(m) b_1^* .$$

4.6.9. PROPOSITION. *For each $m \in M$, the operators $l(m)$ and $l^*(m)$ are adjoints of each other, i.e. we have*

$$\langle l(m)f_1, f_2\rangle = \langle f_1, l^*(m)f_2\rangle \qquad \text{for all } f_1, f_2 \in \mathcal{F}(M).$$

PROOF. It suffices to check this for $f_1 = m_1 \otimes \cdots \otimes m_k$ and $f_2 = \tilde{m}_2 \otimes \cdots \otimes \tilde{m}_k$ with $k \in \mathbb{N}$ and $m_1, \ldots, m_k, \tilde{m}_2, \ldots, \tilde{m}_k \in M$. In this case we have

$$\begin{aligned}
\langle l(m)f_1, f_2\rangle &= \big\langle \langle m, m_1\rangle m_2 \otimes \cdots \otimes m_k, \tilde{m}_2 \otimes \cdots \otimes \tilde{m}_k\big\rangle \\
&= \Big\langle m_k, \ldots, \big\langle m_3, \langle \langle m, m_1\rangle m_2, \tilde{m}_2\rangle \tilde{m}_3\big\rangle \ldots \tilde{m}_k\Big\rangle \\
&= \Big\langle m_k, \ldots, \big\langle m_3, \langle m_2, \langle m, m_1\rangle^* \tilde{m}_2\rangle \tilde{m}_3\big\rangle \ldots \tilde{m}_k\Big\rangle \\
&= \Big\langle m_k, \ldots, \big\langle m_3, \langle m_2, \langle m_1, m\rangle \tilde{m}_2\rangle \tilde{m}_3\big\rangle \ldots \tilde{m}_k\Big\rangle \\
&= \langle m_1 \otimes m_2 \otimes \cdots \otimes m_k, m \otimes \tilde{m}_2 \otimes \cdots \otimes \tilde{m}_k\rangle \\
&= \langle f_1, l^*(m)f_2\rangle. \quad \square
\end{aligned}$$

Furthermore, for $(M, Ł, \rangle)$ being non-degenerate, we define another class of operators on $\mathcal{F}(M)$, namely so-called preservation or gauge operators $p(T)$ (comp. [Par,Mey] for the corresponding scalar notion in the bosonic Fock space). Such an operator $p(T)$ can be defined for each linear operator $T : M \to M$ which possesses an adjoint.

4.6.10. NOTATION. Let $(M, Ł, \rangle)$ be a Hilbert-B-bimodule. By $\mathcal{L}^a(M, M)$ we denote the operators $T \in \mathcal{L}(M, M)$, $T : M \to M$ $\mathbb{C}$-linear, which have an adjoint, i.e. for which there exists a $T^* \in \mathcal{L}(M, M)$ such that

$$\langle Tm_1, m_2\rangle = \langle m_1, T^*m_2\rangle \qquad \text{for all } m_1, m_2 \in M.$$

That we restrict for the definition of $p(T)$ to the non-degenerate case has the following reason (see, e.g., [Pas,Kas]).

4.6.11. PROPOSITION. *Let $(M, Ł, \rangle)$ be a non-degenerate Hilbert-B-bimodule. Then the following assertions are true.*
1) Each $T \in \mathcal{L}(M, M)$ has at most one adjoint.
2) Each $T \in \mathcal{L}^a(M, M)$ is 'left B-linear', i.e. we have for all $b \in B$ and $m \in M$

$$T(mb) = (Tm)b.$$

PROOF. 1) If we have

$$\langle Tm_1, m_2\rangle = \langle m_1, S_1m_2\rangle = \langle m_1, S_2m_2\rangle$$

for all $m_1, m_2 \in M$, then putting $m_1 := S_1m_2 - S_2m_2$ yields $\langle m_1, m_1\rangle = 0$, thus $S_1m_2 = S_2m_2$ for all $m_2 \in M$.
2) Let $T \in \mathcal{L}^a(M, M)$. Then we have for $b \in B$ and $m_1, m_2 \in M$

$$\begin{aligned}
\langle T(m_1b), m_2\rangle &= \langle m_1b, T^*m_2\rangle \\
&= b^*\langle m_1, T^*m_2\rangle \\
&= b^*\langle Tm_1, m_2\rangle \\
&= \langle (Tm_1)b, m_2\rangle,
\end{aligned}$$

which gives for $m_2 := T(m_1 b) - (Tm_1)b$ the assertion. □

4.6.12. Definition. Let $(M, \text{Ł}, \rangle)$ be a non-degenerate Hilbert-B-bimodule and let $T \in \mathcal{L}^a(M, M)$. Then we define the *preservation operator* (or *gauge operator*) $p(T) : \mathcal{F}(M) \to \mathcal{F}(M)$ by ($b \in B$, $m_1, \dots, m_n \in M$)

$$p(T)b = 0$$
$$p(T)m_1 \otimes m_2 \otimes \cdots \otimes m_n = (Tm_1) \otimes m_2 \otimes \cdots \otimes m_n.$$

This is well-defined because of the left B-linearity of T. Between preservation and creation operators we have the relation ($m \in M$, $T \in \mathcal{L}^a(M, M)$)

$$p(T)l^*(m) = l^*(Tm).$$

4.6.13. Proposition. *Let $(M, \text{Ł}, \rangle)$ be a non-degenerate Hilbert-B-bimodule and let $T \in \mathcal{L}^a(M, M)$ with adjoint T^*. Then $p(T^*)$ is an adjoint of $p(T)$, i.e. we have*

$$\langle p(T)f_1, f_2\rangle = \langle f_1, p(T^*)f_2\rangle \qquad \textit{for all } f_1, f_2 \in \mathcal{F}(M).$$

Proof. It suffices to check this for $f_1 = m_1 \otimes \cdots \otimes m_k$ and $f_2 = \tilde{m}_1 \otimes \cdots \otimes \tilde{m}_k$ with $k \in \mathbb{N}$ and $m_1, \dots, m_k, \tilde{m}_1, \dots, \tilde{m}_k \in M$. In this case we have

$$\begin{aligned} \langle p(T)f_1, f_2\rangle &= \langle (Tm_1) \otimes m_2 \otimes \cdots \otimes m_k, \tilde{m}_1 \otimes \tilde{m}_2 \otimes \cdots \otimes \tilde{m}_k\rangle \\ &= \Big\langle m_k, \dots, \langle m_2, \langle Tm_1, \tilde{m}_1\rangle \tilde{m}_2\rangle \dots \tilde{m}_k\Big\rangle \\ &= \Big\langle m_k, \dots, \langle m_2, \langle m_1, T^*\tilde{m}_1\rangle \tilde{m}_2\rangle \dots \tilde{m}_k\Big\rangle \\ &= \langle m_1 \otimes m_2 \otimes \cdots \otimes m_k, (T^*\tilde{m}_1) \otimes \tilde{m}_2 \otimes \cdots \otimes \tilde{m}_k\rangle \\ &= \langle f_1, p(T^*)f_2\rangle. \quad \square \end{aligned}$$

4.6.14. Notations. Let $(M, \text{Ł}, \rangle)$ be a non-degenerate Hilbert-B-bimodule.
1) We denote by

$$A(M) := B\langle l^*(m), l(m), p(T) \mid m \in M, T \in \mathcal{L}^a(M, M)\rangle \subset \mathcal{L}(\mathcal{F}(M), \mathcal{F}(M))$$

the $*$-algebra over B which is generated by B (as left multiplication operators), by all creation and annihilation operators $l^*(m)$ and $l(m)$, respectively, for all $m \in M$, and by all preservation operators $p(T)$ for all $T \in \mathcal{L}^a(M, M)$.
2) If $(M, \text{Ł}, \rangle)$ is the direct sum of two Hilbert-B-bimodules $(M_1, \text{Ł}, \rangle)$ and $(M_2, \text{Ł}, \rangle)$, i.e. if $M = M_1 \oplus M_2$ and

$$\langle m_1 + m_2, \tilde{m}_1 + \tilde{m}_2\rangle = \langle m_1, \tilde{m}_1\rangle + \langle m_2, \tilde{m}_2\rangle$$

for all $m_1, \tilde{m}_1 \in M_1$ and all $m_2, \tilde{m}_2 \in M_2$, then we write

$$(M, \text{Ł}, \rangle) = (M_1, \text{Ł}, \rangle) \oplus (M_2, \text{Ł}, \rangle).$$

In this situation, we will identify an operator $T \in \mathcal{L}^a(M_1, M_1)$ with

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}^a(M_1 \oplus M_2, M_1 \oplus M_2)$$

and hence $A(M_1)$ (and $A(M_2)$ in the same way) can be identified with a subalgebra of $A(M)$,

$$A(M_1), A(M_2) \subset A(M).$$

4.6.15. THEOREM. *Let $(M, \langle\, , \rangle)$ be a non-degenerate Hilbert-B-bimodule. Denote by ϵ_M the 'vacuum expectation'*

$$\begin{aligned}\epsilon_M : A(M) &\to B \\ a &\mapsto \langle 1, a1\rangle.\end{aligned}$$

Then the following statements are true.
1) The vacuum expectation ϵ_M is a positive B-functional.
2) If $(M, \langle\, , \rangle)$ is the direct sum of two Hilbert-B-bimodules $(M_1, \langle\, , \rangle)$ and $(M_2, \langle\, , \rangle)$,

$$(M, \langle\, , \rangle) = (M_1, \langle\, , \rangle) \oplus (M_2, \langle\, , \rangle),$$

then $A(M_1)$ and $A(M_2)$ are free in $(A(M), \epsilon_M)$.

PROOF. 1) This follows directly by Prop. 4.6.4.
2) Consider $a_1, \dots, a_n \in A(M)$ with $a_i \in A(M_{k(i)})$, $k(i) \in \{1, 2\}$, $k(1) \neq k(2) \neq \dots \neq k(n)$, and $\epsilon_M(a_i) = 0$ for all $i = 1, \dots, n$. Then we have to show that $\epsilon_M(a_n \dots a_1) = 0$.
Since $\epsilon_M(a_1) = \langle 1, a_1 1\rangle = 0$, we have

$$a_1 1 \in \bigoplus_{j_1 \geq 1} M_{k(1)}^{\otimes_B j_1} \subset \mathcal{F}(M_1 \oplus M_2).$$

Because of $\epsilon_M(a_2) = 0$ and $k(2) \neq k(1)$ we obtain

$$a_2 a_1 1 \in \Big(\bigoplus_{j_2 \geq 1} M_{k(2)}^{\otimes_B j_2}\Big) \otimes_B \Big(\bigoplus_{j_1 \geq 1} M_{k(1)}^{\otimes_B j_1}\Big),$$

and so on until we end up with

$$a_n \dots a_1 1 \in \Big(\bigoplus_{j_n \geq 1} M_{k(n)}^{\otimes_B j_n}\Big) \otimes_B \dots \otimes_B \Big(\bigoplus_{j_1 \geq 1} M_{k(1)}^{\otimes_B j_1}\Big).$$

But this means in particular that $a_n \dots a_1 1$ does not contain a direct summand $b \in B$ with $b \neq 0$. This gives

$$\epsilon_M(a_n \dots a_1) = \langle 1, a_n \dots a_1 1\rangle = 0. \quad \square$$

4.7. Realization of infinitely divisible distributions on a full Fock space

Let $\mu \in \Sigma_B^{(m)+}$ be a fixed infinitely divisible distribution. Then we want to realize μ as the distribution of a random variable which is the sum of creation, annihilation, and preservation operators on some full Fock space $\mathcal{F}(M)$. More generally, if $\{\mu_t \mid t \geq 0\}$ is the semi-group belonging to μ according to Prop. 4.5.3, then we shall realize a whole process $\{a_t \mid t \geq 0\}$, such that the distribution of a_t is μ_t and that the increments of the process are free.

This construction is motivated by the analogous results of Schürmann [Sch1] for the bosonic case. The special scalar-valued case of the following construction was dealt with in [GSS].

4.7.1. Theorem. *Let $\mu \in \Sigma_B^{(m)+}$ be an infinitely divisible distribution with corresponding semi-group $\{\mu_t \mid t \geq 0\}$. Then there is a canonical construction of a non-degenerate Hilbert-B-bimodule $(\hat{M}, \langle\, ,\, \rangle)$ and operators $a_t^{(i)} = a_t^{(i)*} \in A(\hat{M})$ $(i = 1, \dots, m,\ t \geq 0)$, such that*

i) *for all $t \geq 0$ the joint distribution of $a_t^{(1)}, \dots, a_t^{(m)} \in (A(\hat{M}), \epsilon_{\hat{M}})$ is μ_t.*
ii) *the increments of the process $\{a_t \mid t \geq 0\}$ are free: if we denote by*

$$a_{s,t}^{(i)} := a_t^{(i)} - a_s^{(i)} \qquad (i = 1, \dots, m; t \geq s \geq 0)$$

the increments of $a_t^{(i)}$, then the sets $\{a_{s,t}^{(i)} \mid i = 1, \dots, m\}$ and $\{a_{u,v}^{(i)} \mid i = 1, \dots, m\}$ are free in $(A(\hat{M}), \epsilon_{\hat{M}})$ for $(s,t) \cap (u,v) = \emptyset$.

Proof. We put (with $\mathcal{X}_m = \text{span}\{X_1, \dots, X_m\}$)

$$M_0 := B\langle X_1, \dots, X_m\rangle_0 = B\mathcal{X}_m B \oplus B\mathcal{X}_m B\mathcal{X}_m B \oplus B\mathcal{X}_m B\mathcal{X}_m B\mathcal{X}_m B \oplus \dots,$$

i.e. M_0 are those polynomials in $B\langle \mathcal{X}_m\rangle = B\langle X_1, \dots, X_m\rangle$ which have no constant term. Note that, with $X_i^* = X_i$ for all $i = 1, \dots, m$, the B-B-bimodule M_0 is a $*$-algebra without unit. We define now a B-B-bimodule map

$$\xi : M_0 \to B$$

by

$$\xi(m) := \lim_{t \to 0} \frac{1}{t} \mu_t(m) \qquad \text{for all } m \in M_0 \subset B\langle \mathcal{X}_m\rangle,$$

which, by Prop. 4.5.4, exists and gives nothing else than the cumulants of μ. This ξ is used for defining a B-valued inner product $\langle\, ,\, \rangle_0$ on M_0, namely for $m_1, m_2 \in M_0$ we put

$$\langle m_1, m_2\rangle_0 := \xi(m_1^* m_2) = \lim_{t \to 0} \frac{1}{t} \mu_t(m_1^* m_2).$$

The positivity of $\langle\, ,\, \rangle_0$ is clear, since $\mu_t \in \Sigma_B^{(m)+}$ for all $t \geq 0$ implies that

$$\langle m, m\rangle_0 = \lim_{t \to 0} \frac{1}{t} \mu_t(m^* m) \in B$$

is positive for all $m \in M_0$. Thus $(M_0, \langle\, ,\, \rangle_0)$ is a Hilbert-B-bimodule. Since it may be degenerate in general we have to divide out the null space

$$N := \{n \in M_0 \mid \langle n, n\rangle_0 = 0\} \subset M_0.$$

This gives a non-degenerate Hilbert-B-bimodule $(M, \langle\, ,\, \rangle)$ (comp. Remark 4.6.2). We will denote an element in M_0 and its equivalence class in M by the same symbol. Furthermore, each $T \in \mathcal{L}(M_0, M_0)$ with the property $TN \subset N$ can also be considered as an operator on M, i.e. $T \in \mathcal{L}(M, M)$. In particular, if $\lambda(X_i)$ is the left multiplication operator with X_i, i.e. $(n \in \mathbb{N}_0,\ i_0, \dots, i_n \in \{1, \dots, m\},$ $b_0, \dots, b_{n+1} \in B)$

$$\lambda(X_i) b_0 X_{i_0} b_1 \dots X_{i_n} b_{n+1} = X_i b_0 X_{i_0} b_1 \dots X_{i_n} b_{n+1},$$

then Remark 4.6.2 ensures that $\lambda(X_i)N \subset N$ and hence we have $\lambda(X_i) \in \mathcal{L}(M, M)$. We take now as our basic Hilbert-B-bimodule

$$\hat{M} := L^2(\mathbb{R}_+) \otimes M$$

with ($b_1, b_2 \in B$, $g \in L^2(\mathbb{R}_+)$, $m \in M$)

$$b_1(g \otimes m)b_2 = g \otimes (b_1 m b_2)$$

and ($g_1, g_2 \in L^2(\mathbb{R}_+)$, $m_1, m_2 \in M$)

$$\langle g_1 \otimes m_1, g_2 \otimes m_2 \rangle = \langle g_1, g_2 \rangle \langle m_1, m_2 \rangle.$$

The pair $(\hat{M}, \text{Ł}, \rangle)$ is non-degenerate and we define now our wanted operators by ($i = 1, \dots, m$, $t \geq 0$)

$$a_t^{(i)} := t\mu(X_i) + l(\chi_{(0,t)} \otimes X_i) + l^*(\chi_{(0,t)} \otimes X_i) + p\big(\lambda(\chi_{(0,t)}) \otimes \lambda(X_i)\big) \in A(\hat{M}),$$

where $\chi_{(0,t)} \in L^2(\mathbb{R}_+)$ is the characteristic function of the interval $(0, t) \subset \mathbb{R}_+$ and where $\lambda(\chi_{(0,t)})$ and $\lambda(X_i)$ are the left multiplication operators with $\chi_{(0,t)}$ and X_i on $L^2(\mathbb{R}_+)$ and M, respectively.
The increments of this process are

$$\begin{aligned} a_{s,t}^{(i)} &= (t - s)\mu(X_i) + l(\chi_{(s,t)} \otimes X_i) + l^*(\chi_{(s,t)} \otimes X_i) + p\big(\lambda(\chi_{(s,t)}) \otimes \lambda(X_i)\big) \\ &\in A(L^2(s,t) \otimes M), \end{aligned}$$

and thus, by Theorem 4.6.15, the sets $\{a_{s,t}^{(i)} \mid i = 1, \dots, m\}$ and $\{a_{u,v}^{(i)} \mid i = 1, \dots, m\}$ are free in $(A(\hat{M}), \epsilon_{\hat{M}})$ for $(s,t) \cap (u,v) = \emptyset$. Let us denote, for $0 \leq s \leq t$, the joint distribution of $a_{s,t}^{(1)}, \dots, a_{s,t}^{(m)} \in (A(\hat{M}), \epsilon_{\hat{M}})$ by

$$\tilde{\mu}_{s,t} := \nu_{(a_{s,t}^{(1)}, \dots, a_{s,t}^{(m)})} \in \Sigma_B^{(m)+}.$$

We have to show that $\tilde{\mu}_{0,t} = \mu_t$ for all $t \geq 0$. Because of $\tilde{\mu}_{s,t} \boxplus \tilde{\mu}_{t,v} = \tilde{\mu}_{s,v}$ ($0 \leq s \leq t \leq v$) and $\tilde{\mu}_{s,t} = \tilde{\mu}_{s+v,t+v}$ ($0 \leq s \leq t$, $v \geq 0$), it is clear that $\tilde{\mu} := \tilde{\mu}_{0,1}$ is infinitely divisible and that $\{\tilde{\mu}_{0,t} \mid t \geq 0\}$ is the corresponding semi-group according to Prop. 4.5.3. So we are done if we can show that $\tilde{\mu} = \mu$. For this it suffices to prove the equality of the respective cumulants. By Prop. 4.5.4, we have for all $n \in \mathbb{N}_0$, $i_0, \dots, i_n \in \{1, \dots, m\}$, and $b_1, \dots, b_n \in B$

$$\begin{aligned} \xi_{n;i_0,\dots,i_n}^{(\tilde{\mu})}(b_1, \dots, b_n) &= \lim_{t \to 0} \frac{1}{t} \tilde{\mu}_{0,t}(X_{i_0} b_1 X_{i_1} \dots b_n X_{i_n}) \\ &= \lim_{t \to 0} \frac{1}{t} \epsilon_{\hat{M}}(a_t^{(i_0)} b_1 a_t^{(i_1)} \dots b_n a_t^{(i_n)}) \\ &= \lim_{t \to 0} \frac{1}{t} \langle 1, a_t^{(i_0)} b_1 a_t^{(i_1)} \dots b_n a_t^{(i_n)} 1 \rangle, \end{aligned}$$

whereas the cumulants of μ are, by definition of ξ, nothing else than

$$\xi_{n;i_0,\dots,i_n}^{(\mu)}(b_1, \dots, b_n) = \xi(X_{i_0} b_1 X_{i_1} \dots b_n X_{i_n}).$$

We consider now the three case $n = 0$, $n = 1$, and $n \geq 2$ separately.
First, for $n = 0$ we have

$$\begin{aligned}\xi^{(\tilde{\mu})}_{0;i}(1) &= \lim_{t\to 0} \frac{1}{t}\langle 1, a_t^{(i)} 1\rangle \\ &= \lim_{t\to 0} \frac{1}{t}\big\{ t\mu(X_i)\big\} \\ &= \mu(X_i) \\ &= \xi^{(\mu)}_{0;i}(1).\end{aligned}$$

For $n = 1$ we have

$$\begin{aligned}\xi^{(\tilde{\mu})}_{1;i,j}(b) &= \lim_{t\to 0} \frac{1}{t}\langle 1, a_t^{(i)} b a_t^{(j)} 1\rangle \\ &= \lim_{t\to 0} \frac{1}{t}\big\{\langle 1, l(\chi_{(0,t)} \otimes X_i) b l^*(\chi_{(0,t)} \otimes X_j)1\rangle + O(t^2)\big\} \\ &= \lim_{t\to 0} \frac{1}{t}\big\{\langle \chi_{(0,t)}, \chi_{(0,t)}\rangle \langle X_i, bX_j\rangle\big\} \\ &= \xi(X_i b X_j) \\ &= \xi^{(\mu)}_{1;i,j}(b).\end{aligned}$$

Finally, the case $n \geq 2$ gives

$$\begin{aligned}\xi^{(\tilde{\mu})}_{n;i_0,\dots,i_n}(b_1,\dots,b_n) &= \lim_{t\to 0} \frac{1}{t}\langle 1, a_t^{(i_0)} b_1 a_t^{(i_1)} \dots a_t^{(i_{n-1})} b_n a_t^{(i_n)} 1\rangle \\ &= \lim_{t\to 0} \frac{1}{t}\big\{\langle 1, l(\chi_{(0,t)} \otimes X_{i_0}) b_1 p\big(\lambda(\chi_{(0,t)}) \otimes \lambda(X_{i_1})\big) \dots \\ &\qquad \dots p\big(\lambda(\chi_{(0,t)}) \otimes \lambda(X_{i_{n-1}})\big) b_n l^*(\chi_{(0,t)} \otimes X_{i_n})1\rangle + O(t^2)\big\} \\ &= \langle X_{i_0}, b_1 X_{i_1} \dots b_{n-1} X_{i_{n-1}} b_n X_{i_n}\rangle \\ &= \xi(X_{i_0} b_1 X_{i_1} \dots b_n X_{i_n}) \\ &= \xi^{(\mu)}_{n;i_0,\dots,i_n}(b_1,\dots,b_n).\end{aligned}$$

Thus, in any case, the cumulants of μ and of $\tilde{\mu}$ are the same and hence we have $\mu = \tilde{\mu}$. □

4.7.2. Remark. It seems conceivable that the more general results of Schürmann [ASW,Sch2,Sch3] for white noises on arbitrary bialgebras possess a similar free counterpart after replacing the notion of 'bialgebra' by the concept of 'dual group' [Voi6].