CHAPTER III

AMALGAMATED FREE PRODUCTS

In this chapter we will define the notion of the amalgamated free product. This (reduced) free product was introduced by Voiculescu [Voi1] as a generalization of his scalar-valued free product. In [Spe4], we realized that the lattice of non-crossing partitions governs the structure of the scalar-valued free product. Since a non-crossing partition $\pi \in NC(n)$ corresponds canonically to a bracketing of a monomial of n factors, the combinatorial description admits a canonical generalization to the operator-valued case.

As in the scalar-valued case the main concepts are the notions of moment and cumulant function, which are quite special multiplicative functions and which are related by convolution with the zeta or Möbius function. The philosophy behind our combinatorial approach is that the free product is linearized by the cumulant functions.

3.1. Basic notations

In the following B will be a fixed unital algebra. Then we will work in the category of algebras over B and of B-functionals.

3.1.1. Definition. Let B be a unital algebra.
1) A unital algebra A with $1 \in B \subset A$ is called an *algebra over* B.
2) Given an algebra A over B, we call a linear mapping $\varphi : A \to B$ a *B-functional* (or *conditional expectation*) if it is the identity on B,

$$\varphi(b) = b \qquad \text{for all } b \in B \qquad (\text{in particular } \varphi(1) = 1),$$

and if it has the bimodule property

$$\varphi(b_1 a b_2) = b_1 \varphi(a) b_2 \qquad \text{for all } b_1, b_2 \in B,\ a \in A.$$

3) If B is commutative, then we call a B-functional $\varphi : A \to B$ a *trace*, if we have

$$\varphi(a_1 \dots a_n) = \varphi(a_n a_1 \dots a_{n-1}) \qquad \text{for all } n \in \mathbb{N} \text{ and } a_1, \dots, a_n \in A.$$

Given algebras A_i over B $(i \in I)$ with corresponding B-functionals $\varphi_i : A_i \to B$, we want to define the amalgamated free product $*_{i\in I}\varphi_i$ of the φ_i as a B-functional on the algebraic free product

$$A := \mathop{*_B}_{i \in I} A_i \qquad \text{with amalgamation over } B.$$

This algebraic amalgamated free product is given by the free algebra in all elements from all A_i $(i \in I)$, divided by the relations within each A_i for all $i \in I$ and the additional relations which identify all the subsets B from all A_i (see, e.g., [Bou]). In the following we will usually identify A_k with its image under the canonical embedding $A_k \to *_B A_i$.

The idea for the definition of the free product functional φ is the following: Each B-functional φ_i determines in a canonical way a multiplicative moment function $\hat{\varphi}_i$, from which we calculate a cumulant function $\hat{c}_i = \hat{\varphi}_i \star \mu$. Then we use the idea that cumulants linearize our free product and define the cumulant function corresponding to φ as the 'direct sum' of the cumulant functions $\hat{c}_i$. This direct sum can be defined explicitly only on the set

$$\mathring{A} := \bigcup_{i \in I} A_i \subset \underset{i \in I}{*_B} A_i,$$

which is a sub-module of the B-B-bimodule A with $B \subset \mathring{A}$. Hence we shall in general consider cumulant and moment functions which are not defined on the whole algebra A, but only on such submodules $\mathring{A}$ of A as just indicated.

3.2. Moment and cumulant functions

Our setting will now be the following: A is a unital algebra over B and $\mathring{A}$ is a sub-module of the B-B-bimodule A. We also require that $\mathring{A}$ contains 1, and hence B, i.e. we have

$$1 \in B \subset \mathring{A} \subset A, \qquad \mathring{A} \text{ sub-module of } A.$$

3.2.1. Definition. 1) A multiplicative $\hat{\varphi} = (\varphi^{(n)}) \in \mathbf{I}(\mathring{A}, B)$ is called *moment function*, if we have

$$\varphi^{(1)}(1) = 1 \qquad (\text{thus } \varphi^{(1)}(b) = b \text{ for all } b \in B)$$

and

$$\varphi^{(n-1)}(a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n) = \varphi^{(n)}(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n)$$

for all $n \geq 2$, all $1 \leq p \leq n-1$, and all $a_1, \dots, a_n \in \mathring{A}$ with $a_p a_{p+1} \in \mathring{A}$. The set of all such multiplicative moment functions will be denoted by $\mathbf{I}^m(\mathring{A}, B)$.
2) A multiplicative $\hat{c} = (c^{(n)}) \in \mathbf{I}(\mathring{A}, B)$ is called *cumulant function*, if we have

$$c^{(1)}(1) = 1 \qquad (\text{thus } c^{(1)}(b) = b \text{ for all } b \in B)$$

and

$$\begin{aligned} c^{(n-1)}(a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n) &= c^{(n)}(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n) + \\ &+ \sum_{\substack{\pi \in NC(n) \\ |\pi| = 2 \\ p \not\sim_\pi p+1}} \hat{c}(\pi)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n]. \end{aligned}$$

for all $n \geq 2$, all $1 \leq p \leq n-1$, and all $a_1, \dots, a_n \in \mathring{A}$ with $a_p a_{p+1} \in \mathring{A}$. The set of all such multiplicative cumulant functions will be denoted by $\mathbf{I}^c(\mathring{A}, B)$.

3.2.2. Remarks. 1) The cumulant property of $\hat{c} \in \mathbf{I}^c(\mathring{A}, B)$ may also be stated as ($\pi \in NC(n-1)$, $a_1, \dots, a_n, a_p a_{p+1} \in \mathring{A}$)

$$\hat{c}(\pi)[a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n] = \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \pi}} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n],$$

where $\sigma|_{p=p+1} \in NC(n-1)$ denotes that partition which results from σ by identifying the points p and $p+1$, i.e. if $p \sim_\sigma p+1$, then we just remove $p+1$, whereas for $p \not\sim_\sigma p+1$ we also have to merge the two blocks containing p and $p+1$, respectively, to one block. Note that, of course,

$$\sum_{\pi \in NC(n-1)} \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \pi}} = \sum_{\sigma \in NC(n)} .$$

2) The corresponding general version for $\hat{\varphi} \in \mathbf{I}^m(\mathring{A}, B)$ is

$$\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n] = \hat{f}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n],$$

where $\sigma \in NC(n)$ is uniquely determined by $\sigma|_{p=p+1} = \pi$ and $p \sim_\sigma p+1$.

3.2.3. Proposition. *Consider $\hat{\varphi}, \hat{c} \in \mathbf{I}(\mathring{A}, B)$ which are related according to*

$$\hat{\varphi} = \hat{c} \star \zeta \qquad or \qquad \hat{c} = \hat{\varphi} \star \mu.$$

Then, $\hat{\varphi}$ is a moment function, if and only if $\hat{c}$ is a cumulant function.

Formally, we may write

$$\mathbf{I}^m = \mathbf{I}^c \star \zeta \qquad \text{and} \qquad \mathbf{I}^c = \mathbf{I}^m \star \mu.$$

Proof. First, we show that $\hat{c} \in \mathbf{I}^c(\mathring{A}, B)$ implies $\hat{\varphi} \in \mathbf{I}^m(\mathring{A}, B)$. So assume that $\hat{c}$ is a cumulant function. Then

$$\begin{aligned}
\varphi^{(n-1)}(a_1 \otimes \cdots \otimes & a_p a_{p+1} \otimes \cdots \otimes a_n) = \\
&= \sum_{\pi \in NC(n-1)} \hat{c}(\pi)[a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n] \\
&= \sum_{\pi \in NC(n-1)} \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \pi}} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n] \\
&= \sum_{\sigma \in NC(n)} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n] \\
&= \varphi^{(n)}(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n),
\end{aligned}$$

i.e. $\hat{\varphi} \in \mathbf{I}^m(\mathring{A}, B)$.
Assume now that $\hat{\varphi} \in \mathbf{I}^m(\mathring{A}, B)$ is a moment function. We have to show that

$\hat{c} = (c^{(n)})$ is a cumulant function. The cumulant property of $c^{(n)}$ will be shown by induction on n. For $n = 2$, the assertion is clear, since

$$\begin{aligned} c^{(1)}(a_1 a_2) &= \varphi^{(1)}(a_1 a_2) \\ &= \varphi^{(2)}(a_1 \otimes a_2) \\ &= c^{(2)}(a_1 \otimes a_2) + c^{(1)}(a_1) \cdot c^{(1)}(a_2) \\ &= c^{(2)}(a_1 \otimes a_2) + \hat{c}\big((1),(2)\big)[a_1 \otimes a_2]. \end{aligned}$$

Now consider $n \geq 3$ and assume the cumulant property to be true for all $n' < n$, in particular

$$\hat{c}(\pi)[a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n] = \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \pi}} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n]$$

for all $\pi \in NC(n-1)$ with $\pi \neq \mathbf{1}_{n-1}$. Then we have

$$\begin{aligned} & c^{(n-1)}(a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n) = \\ &\quad = \varphi^{(n-1)}(a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n) \\ &\qquad - \sum_{\substack{\pi \in NC(n-1) \\ \pi \neq \mathbf{1}_{n-1}}} \hat{c}(\pi)[a_1 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n] \\ &\quad = \varphi^{(n)}(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n) \\ &\qquad - \sum_{\substack{\pi \in NC(n-1) \\ \pi \neq \mathbf{1}_{n-1}}} \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \pi}} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n] \\ &\quad = \varphi^{(n)}(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n) \\ &\qquad - \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} \neq \mathbf{1}_{n-1}}} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n] \\ &\quad = \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \mathbf{1}_{n-1}}} \hat{c}(\sigma)[a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n], \end{aligned}$$

which is the cumulant property for n. $\square$

3.2.4. Lemma. *Let $\hat{c} = (c^{(n)}) \in \mathbf{I}^c(\mathring{A}, B)$ be a cumulant function. Then, for $n \geq 2$ and $a_1, \ldots, a_n \in \mathring{A}$,*

$$c^{(n)}(a_1 \otimes \cdots \otimes a_n) = 0, \quad \textit{if there exists } p \in \{1, \ldots, n\} \textit{ such that } a_p \in B.$$

Proof. By induction on n. For $n = 2$ and $a_2 = b \in B$ we have

$$c^{(1)}(a_1)b = c^{(1)}(a_1 b) = c^{(2)}(a_1 \otimes b) + c^{(1)}(a_1) \cdot c^{(1)}(b) = c^{(2)}(a_1 \otimes b) + c^{(1)}(a_1)b,$$

implying $c^{(2)}(a_1 \otimes b) = 0$. The case $a_1 = b \in B$ is similar.
Now assume the assertion to be true for all $n' < n$, where $n \geq 3$. We consider the

case $p \in \{1, \dots, n-1\}$, $p = n$ is analogous. We have with $a_p = b \in B$ (note that $ba_{p+1} \in \mathring{A}$)

$$c^{(n)}(a_1 \otimes \cdots \otimes b \otimes a_{p+1} \otimes \cdots \otimes a_n) = c^{(n-1)}(a_1 \otimes \cdots \otimes ba_{p+1} \otimes \cdots \otimes a_n) - \sum_{\substack{\pi \in NC(n) \\ |\pi|=2 \\ p \not\sim_\pi p+1}} \hat{c}(\pi)[a_1 \otimes \cdots \otimes b \otimes a_{p+1} \otimes \cdots \otimes a_n].$$

Because of our induction hypothesis only $\pi_0 := \{(p), (1, \dots, p-1, p+1, \dots, n)\} \in NC(n)$ gives a non-vanishing contribution in the sum, i.e.

$$\begin{aligned} &c^{(n)}(a_1 \otimes \cdots \otimes b \otimes a_{p+1} \otimes \cdots \otimes a_n) = \\ &= c^{(n-1)}(a_1 \otimes \cdots \otimes ba_{p+1} \otimes \cdots \otimes a_n) - \hat{c}(\pi_0)[a_1 \otimes \cdots \otimes b \otimes a_{p+1} \otimes \cdots \otimes a_n] \\ &= c^{(n-1)}(a_1 \otimes \cdots \otimes ba_{p+1} \otimes \cdots \otimes a_n) - c^{(n-1)}\big(a_1 \otimes \cdots \otimes c^{(1)}(b)a_{p+1} \otimes \cdots \otimes a_n\big) \\ &= 0. \quad \square \end{aligned}$$

3.2.5. Proposition. *If $\mathring{A} = A$ is an algebra over B, then each moment function $\hat{\varphi} = (\varphi^{(n)}) \in \mathbf{I}^m(A, B)$ is uniquely determined by a B-functional $\varphi : A \to B$ via*

$$\varphi^{(n)}(a_1 \otimes \cdots \otimes a_n) = \varphi(a_1 \dots a_n) \qquad (n \in \mathbb{N}, a_1, \dots, a_n \in A).$$

Proof. Given a B-functional $\varphi : A \to B$, we define $\hat{\varphi} = (\varphi^{(n)})$ by the above equation. Of course, $\hat{\varphi}$ is a moment function.
Conversely, given $\hat{\varphi} = (\varphi^{(n)}) \in \mathbf{I}^m(A, B)$ we put $\varphi := \varphi^{(1)}$. Because of $\varphi^{(1)}(1) = 1$ and the bimodule property of the $\varphi^{(n)}$ we have $\varphi(b) = \varphi^{(1)}(b) = b$ for $b \in B$, thus φ is indeed a B-functional. Since $\hat{\varphi} \in \mathbf{I}^m(A, B)$, we get the required equation, namely

$$\varphi^{(n)}(a_1 \otimes \cdots \otimes a_n) = \varphi^{(1)}(a_1 \dots a_n) = \varphi(a_1 \dots a_n). \quad \square$$

3.2.6. Notation. In the situation considered in Prop. 3.2.5 we will say that $\hat{\varphi}$ is the *moment function* of $\varphi^{(1)} = \varphi$ and that the corresponding $\hat{c} = \hat{\varphi} \star \mu$ is the *cumulant function* of $c^{(1)} = \varphi^{(1)} = \varphi$.

3.2.7. Examples. 1) For illustration, let us write down for this case the explicit formulas for small n for the connection between the B-functional $\varphi : A \to B$ and its moment function $\hat{\varphi} = (\varphi^{(n)})$ and its cumulant function $\hat{c} = (c^{(n)})$:

$$\begin{aligned} c^{(1)}(a_1) &= \varphi^{(1)}(a_1) \\ &= \varphi(a_1) \\ c^{(2)}(a_1 \otimes a_2) &= \varphi^{(2)}(a_1 \otimes a_2) - \varphi^{(1)}(a_1) \cdot \varphi^{(1)}(a_2) \\ &= \varphi(a_1 a_2) - \varphi(a_1) \cdot \varphi(a_2) \\ c^{(3)}(a_1 \otimes a_2 \otimes a_3) &= \varphi^{(3)}(a_1 \otimes a_2 \otimes a_3) - \varphi^{(1)}(a_1) \cdot \varphi^{(2)}(a_2 \otimes a_3) \\ &\quad - \varphi^{(2)}\big(a_1 \varphi^{(1)}(a_2) \otimes a_3\big) - \varphi^{(2)}(a_1 \otimes a_2) \cdot \varphi^{(1)}(a_3) \\ &\quad + 2\varphi^{(1)}(a_1) \cdot \varphi^{(1)}(a_2) \cdot \varphi^{(1)}(a_3) \\ &= \varphi(a_1 a_2 a_3) - \varphi(a_1) \cdot \varphi(a_2 a_3) - \varphi\big(a_1 \varphi(a_2) a_3\big) \\ &\quad - \varphi(a_1 a_2) \cdot \varphi(a_3) + 2\varphi(a_1) \cdot \varphi(a_2) \cdot \varphi(a_3). \end{aligned}$$

2) Note that the function $\hat{\varphi}(\pi)$ is now nothing else than the evaluation with the help of φ of the bracketing corresponding to π. Our example 2.1.2 reads for the moment function as follows

$$\hat{\varphi}\big((1,6,8),(2,5),(3,4),(7),(9)\big)[a_1 \otimes \cdots \otimes a_9] = \\ = \varphi\Big(a_1\varphi\big(a_2\varphi(a_3a_4)a_5\big)a_6\varphi(a_7)a_8\Big)\varphi(a_9)$$

corresponding to the bracketing

$$\Big(a_1\big(a_2(a_3a_4)a_5\big)a_6(a_7)a_8\Big)(a_9).$$

3.2.8. PROPOSITION. *Consider the special case*

$$\mathring{A} := \bigcup_{i\in I} A_i \subset \mathop{*_B}_{i\in I} A_i =: A.$$

Then each moment function $\hat{\varphi} \in \mathbf{I}^m(\mathring{A}, B)$ is the restriction of a uniquely determined moment function $\hat{\phi} \in \mathbf{I}^m(A,B)$. In particular, $\hat{\varphi}$ is the restriction of the moment function of a uniquely determined B-functional $\varphi : A \to B$.

PROOF. Let $\hat{\varphi} = (\varphi^{(n)}) \in \mathbf{I}^m(\mathring{A}, B)$ be a moment function. Each element a in $A = *_B A_i$ can be written as a sum of elements $a = a_1 \dots a_n$ with $n \in \mathbb{N}$ and $a_i \in \mathring{A}$. We put $\varphi(a) := \varphi^{(n)}(a_1 \otimes \cdots \otimes a_n)$. Of course, the representation of a is not unique, but the only relations are the identification of all $B \subset A_i$, the relations within all A_i and linear relations. Since $\hat{\varphi}$ is a moment function on $\mathring{A} = \bigcup_{i\in I} A_i$, it respects all these universal relations of the free product, thus the definition of $\varphi(a)$ is independent of the special choice for the representation of a. Hence $\varphi : A \to B$ is a well-defined B-functional, which gives rise to the corresponding moment function $\hat{\phi} = (\phi^{(n)}) \in \mathbf{I}^m(A,B)$ via

$$\phi^{(n)}(a_1 \otimes \cdots \otimes a_n) = \varphi(a_1 \dots a_n) \qquad (n \in \mathbb{N},\ a_1, \dots, a_n \in A).$$

Of course, $\hat{\phi}$ is an extension of $\hat{\varphi}$, and $\hat{\phi}$ and φ are uniquely determined by $\hat{\varphi}$. □

3.3. Definition of the amalgamated free product

Now we can define our amalgamated free product. We are given B-functionals $\varphi_i : A_i \to B$ $(i \in I)$ and we want to define the B-functional

$$*_{i\in I}\varphi_i : \mathop{*_B}_{i\in I} A_i \to B.$$

Each φ_i $(i \in I)$ determines, by Prop. 3.2.5, a moment function $\hat{\varphi}_i = (\varphi_i^{(n)}) \in \mathbf{I}^m(A_i, B)$ and a corresponding cumulant function $\hat{\varphi}_i \star \mu =: \hat{c}_i = (c_i^{(n)}) \in \mathbf{I}^c(A_i, B)$. According to the idea that cumulants linearize the free product we define

$$\hat{c} := \bigoplus_{i\in I} \hat{c}_i \qquad \text{on} \qquad \mathring{A} := \bigcup_{i\in I} A_i \subset \mathop{*_B}_{i\in I} A_i$$

by $\hat{c} = (c^{(n)})$ with $(n \in \mathbb{N},\ a_1, \dots, a_n \in \mathring{A})$

$$c^{(n)}(a_1 \otimes \cdots \otimes a_n) = \\ = \begin{cases} c_k^{(n)}(a_1 \otimes \cdots \otimes a_n), & \text{if } a_1, \dots, a_n \in A_k \text{ for a } k \in I \\ 0, & \text{otherwise.} \end{cases}$$

Note that the only ambiguity in this definition might appear if at least one of the a_i is in $A_k \cap A_l = B$. But in this case all possible cases of the definition agree, namely, for $n = 1$, we have $c^{(1)}(b) = c_k^{(1)}(b) = c_l^{(1)}(b) = b$ and, for $n \geq 2$, we have in any case, by Lemma 3.2.4, that $c^{(n)}(a_1 \otimes \cdots \otimes a_n) = 0$. Thus $\hat{c} \in \mathbf{I}(\mathring{A}, B)$ is well-defined. It is also easy to check that the cumulant property for all $\hat{c}_i$ implies this property also for $\hat{c}$, thus $\hat{c} \in \mathbf{I}^c(\mathring{A}, B)$. But then $\hat{c} \star \zeta =: \hat{\varphi} \in \mathbf{I}^m(\mathring{A}, B)$ is, by Prop. 3.2.8, the restriction of the moment function of a uniquely determined B-functional $\varphi : *_B A_i \to B$. This φ is our amalgamated free product of the φ_i.

3.3.1. Definition. 1) Let, for $i \in I$, $\varphi_i : A_i \to B$ be B-functionals with cumulant functions $\hat{c}_i \in \mathbf{I}^c(A_i, B)$. The B-functional

$$\varphi = *_{i \in I} \varphi_i : \underset{i \in I}{*_B} A_i \to B,$$

uniquely determined by the requirement that the restriction to $\mathring{A} := \bigcup_{i \in I} A_i$ of its cumulant function coincides with $\bigoplus_{i \in I} \hat{c}_i \in \mathbf{I}^c(\mathring{A}, B)$, is called the *amalgamated free product* of the φ_i.
2) Let an algebra A over B and a B-functional $\varphi : A \to B$ be given and let A_i $(i \in I)$ be subalgebras of A with $B \subset A_i \subset A$. Denote by $\hat{c}_i \in \mathbf{I}^c(A_i, B)$ the restriction of the cumulant function of φ to A_i. If the restriction of the cumulant function of φ to $\mathring{A} := \bigcup_{i \in I} A_i$ coincides with $\bigoplus_{i \in I} \hat{c}_i \in \mathbf{I}^c(\mathring{A}, B)$, then we say that the family $(A_i)_{i \in I}$ *is free in* (A, φ).
3) Let an algebra A over B and a B-functional $\varphi : A \to B$ be given and let X_i $(i \in I)$ be subsets of A. Denote, for each $i \in I$, by $A_i \subset A$ the algebra generated by B and X_i. Then we say that the family $(X_i)_{i \in I}$ *is free in* (A, φ) if the corresponding family $(A_i)_{i \in I}$ is free in (A, φ).

3.3.2. Remarks. 1) Note that our operation $*$ is clearly associative, since, e.g., $(\varphi_1 * \varphi_2) * \varphi_3$ and $\varphi_1 * (\varphi_2 * \varphi_3)$ are both B-functionals on $(A_1 *_B A_2) *_B A_3 = A_1 *_B A_2 *_B A_3 = A_1 *_B (A_2 *_B A_3)$, and the restrictions to $A_1 \cup A_2 \cup A_3$ of their cumulant functions coincide, so they must be equal.
2) Lemma 3.2.4 implies that if $(A_i)_{i \in I}$ is free in (A, φ), then $B \cup (A_i)_{i \in I}$ is free in (A, φ), too. In particular, we have that B and A are free in (A, φ).

The next proposition shows that our amalgamated free product coincides with the corresponding notion of Voiculescu [Voi1,Voi4].

3.3.3. Proposition. *Let* $\varphi = *_{i \in I} \varphi_i : *_B A_i \to B$.
1) The restriction of φ *to* A_i *is* φ_i $(i \in I)$,

$$\varphi|_{A_i} = \varphi_i \qquad \text{for all } i \in I.$$

2) Let $a_k \in A_{i(k)}$ $(k = 1, \dots, n)$ *such that* $i(k) \neq i(k+1)$ *for* $k = 1, \dots, n-1$ *and* $\varphi(a_k) = \varphi_{i(k)}(a_k) = 0$ *for* $k = 1, \dots, n$. *Then* $\varphi(a_1 \dots a_n) = 0$.

3) Let $a_k \in A_{i(k)}$ $(k = 1, \dots, n)$ and $\tilde{a}_l \in A_{j(l)}$ $(l = 1, \dots, m)$ such that $i(k) \neq i(k+1)$ $(k = 1, \dots, n-1)$ and $j(l) \neq j(l+1)$ $(l = 1, \dots, m-1)$ and $\varphi(a_k) = \varphi(\tilde{a}_l) = 0$ for $k = 1, \dots, n$ and $l = 1, \dots, m$. Then

$$\varphi(a_1 a_2 \dots a_n \tilde{a}_m \dots \tilde{a}_2 \tilde{a}_1) = \delta_{nm} \varphi\Big(a_1 \varphi\big(a_2 \dots \varphi(a_n \tilde{a}_n) \dots \tilde{a}_2\big) \tilde{a}_1\Big).$$

In particular, $\varphi(a_1 \dots a_n \tilde{a}_m \dots \tilde{a}_1)$ is only non-vanishing, if $n = m$ and $i(k) = j(k)$ for all $k = 1, \dots, n$.

PROOF. 1) We have for $a \in A_i$

$$\varphi(a) = \varphi^{(1)}(a) = c^{(1)}(a) = c_i^{(1)}(a) = \varphi_i^{(1)}(a) = \varphi_i(a).$$

2) For each interval $[k, l]$ in $(1, \dots, n)$ we have $c^{(l-k+1)}(a_k \otimes \dots \otimes a_l) = 0$, because (for $l = k$) $c^{(1)}(a_k) = \varphi^{(1)}(a_k) = \varphi(a_k) = 0$ and (for $l > k$) $c^{(\dots)}(a_k \otimes a_{k+1} \otimes \dots) = 0$ by the definition of $\hat{c} = \bigoplus_{i \in I} \hat{c}_i$ (a_k and a_{k+1} are from different algebras). Since each $\pi \in NC(n)$ contains such an interval, we have $\hat{c}(\pi)[a_1 \otimes \dots \otimes a_n] = 0$ for all $\pi \in NC(n)$ and thus

$$\varphi(a_1 \dots a_n) = \varphi^{(n)}(a_1 \otimes \dots \otimes a_n) = \sum_{\pi \in NC(n)} \hat{c}(\pi)[a_1 \otimes \dots \otimes a_n] = 0.$$

3) This follows either directly by the characterization 2) (compare [VDN]), or it may be seen in our frame as follows: We have

$$\begin{aligned} \varphi(a_1 \dots a_n \tilde{a}_m \dots \tilde{a}_1) &= \varphi^{(n+m)}(a_1 \otimes \dots \otimes a_n \otimes \tilde{a}_m \otimes \dots \otimes \tilde{a}_1) \\ &= \sum_{\pi \in NC(n+m)} \hat{c}(\pi)[a_1 \otimes \dots \otimes a_n \otimes \tilde{a}_m \otimes \dots \otimes \tilde{a}_1]. \end{aligned}$$

However only such π contribute in the sum which connect elements from the same algebra and which, by $\varphi(a_k) = \varphi(\tilde{a}_l) = 0$, have no blocks consisting of a single element. A moment's reflection reveals that, for $n \neq m$, there is no such non-crossing partition in $NC(n+m)$, whereas, for $n = m$, there is exactly one such π, namely

$$\pi = \big\{(1, 2n), (2, 2n-1), \dots, (n, n+1)\big\}.$$

This, together with the fact $c^{(2)}(a_k \otimes b\tilde{a}_k) = \varphi(a_k b \tilde{a}_k)$ for $b \in B$, gives the assertion. □

Our general considerations in the last chapter on the connection between $\hat{f}$ and $\hat{g} = \hat{f} \star \zeta$ yield now directly that some properties of the φ_i transfer also to the free product $*_{i \in I} \varphi_i$.

3.3.4. PROPOSITION. *1) If all $\varphi_i : A_i \to B$ $(i \in I)$ are homomorphisms, then $*_{i \in I} \varphi_i : *_B A_i \to B$ is a homomorphism, too.*
2) Let B be commutative and assume that all $\varphi_i : A_i \to B$ $(i \in I)$ are traces. Then $*_{i \in I} \varphi_i : *_B A_i \to B$ is a trace, too.

PROOF. 1) Let, for $i \in I$, $\hat{\varphi}_i = (\varphi^{(n)}) \in \mathbf{I}^m(A_i, B)$ be the moment function of φ_i and $\hat{c}_i = (c_i^{(n)}) := \hat{\varphi}_i \star \mu \in \mathbf{I}^c(A_i, B)$ the corresponding cumulant function. Since φ_i is a homomorphism we have

$$\begin{aligned} \hat{\varphi}_i^{(n)}(a_1 \otimes \dots \otimes a_n) &= \varphi_i(a_1 \dots a_n) \\ &= \varphi_i(a_1) \dots \varphi_i(a_n) \\ &= \varphi_i^{(1)}(a_1) \dots \varphi_i^{(1)}(a_n) \end{aligned}$$

for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in A_i$. But this means for all $n \geq 2$, by Corollary 2.5.5, $c_i^{(n)} \equiv 0$. This implies of course the same property for $\hat{c} = \bigoplus_{i \in I} \hat{c}_i$ and thus, again by Corollary 2.5.5, the homomorphism property for the B-functional corresponding to $\hat{c}$, i.e. for $*_{i \in I} \varphi_i$.
2) The fact that φ_i is a trace is the same as the fact that the corresponding moment function is tracial in the sense of Def. 2.4.1. Since $\bigoplus_{i \in I} \hat{c}_i$ is tracial if the $\hat{c}_i$ are, the assertion follows by Prop. 2.4.3. □

3.3.5. Remark. Note that in the case where all $\varphi_i : A_i \to B$ are homomorphisms our free product coincides with the usual notion of a 'free product of homomorphisms', see, e.g., [Bou].

3.3.6. Examples. Let us illustrate the calculation procedure for $\varphi = *_{i \in I} \varphi_i$ out of the φ_i by some examples. We will always assume that $a_k \in A_{i(k)}$.
1) The (A_i, φ_i) are stochastically independent in (A, φ), which means

$$\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2) \qquad \text{if } i(1) \neq i(2).$$

We even have the stronger pyramidal factorization property (which plays a fundamental role in Kümmerer's stochastic integration theory [KP,Küm1,Küm2,KSp]), namely

$$\varphi(a_1 a_2 a_3) = \varphi\big(a_1 \varphi(a_2) a_3\big) \qquad \text{if } i(1) = i(3) \neq i(2).$$

This can be seen as follows: Only $\pi \in NC(3)$ appear which do not couple $\{2\}$ with $\{1,3\}$, thus, by putting $i(1) := 1$ and $i(2) := 2$,

$$\begin{aligned}
\varphi(a_1 a_2 a_3) &= \varphi^{(3)}(a_1 \otimes a_2 \otimes a_3) \\
&= \hat{c}\big((1),(2),(3)\big)[a_1 \otimes a_2 \otimes a_3] + \hat{c}\big((1,3),(2)\big)[a_1 \otimes a_2 \otimes a_3] \\
&= c_1^{(1)}(a_1) \cdot c_2^{(1)}(a_2) \cdot c_1^{(1)}(a_3) + c_1^{(2)}\big(a_1 \otimes c_2^{(1)}(a_2) a_3\big) \\
&= \varphi_1^{(2)}\big(a_1 \otimes \varphi_2^{(1)}(a_2) a_3\big) \\
&= \varphi\big(a_1 \varphi(a_2) a_3\big).
\end{aligned}$$

2) Assume $1 = i(1) = i(3) \neq i(2) = i(4) = 2$ and consider $\varphi(a_1 a_2 a_3 a_4)$. Only $\pi \in NC(4)$ appear in our formula which do not couple $\{1,3\}$ with $\{2,4\}$, thus

$$\begin{aligned}
\varphi(a_1 a_2 a_3 a_4) &= \varphi^{(4)}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) \\
&= \hat{c}\big((1),(2),(3),(4)\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] + \hat{c}\big((1,3),(2),(4)\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \\
&\quad + \hat{c}\big((1),(2,4),(3)\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \\
&= c_1^{(1)}(a_1) \cdot c_2^{(1)}(a_2) \cdot c_1^{(1)}(a_3) \cdot c_2^{(1)}(a_4) + c_1^{(2)}\big(a_1 \otimes c_2^{(1)}(a_2) a_3\big) \cdot c_2^{(1)}(a_4) \\
&\quad + c_1^{(1)}(a_1) \cdot c_2^{(2)}\big(a_2 \otimes c_1^{(1)}(a_3) a_4\big) \\
&= \varphi(a_1) \cdot \varphi(a_2) \cdot \varphi(a_3) \cdot \varphi(a_4) + \big\{\varphi\big(a_1 \varphi(a_2) a_3\big) - \varphi(a_1) \cdot \varphi(a_2) \cdot \varphi(a_3)\big\}\varphi(a_4) \\
&\quad + \varphi(a_1)\big\{\varphi\big(a_2 \varphi(a_3) a_4\big) - \varphi(a_2) \cdot \varphi(a_3) \cdot \varphi(a_4)\big\} \\
&= \varphi\big(a_1 \varphi(a_2) a_3\big) \cdot \varphi(a_4) + \varphi(a_1) \cdot \varphi\big(a_2 \varphi(a_3) a_4\big) - \varphi(a_1) \cdot \varphi(a_2) \cdot \varphi(a_3) \cdot \varphi(a_4).
\end{aligned}$$

3.3.7. Notation. Sometimes one would also like to take the amalgamated free product of a B-functional $\varphi : A \to B$ with an ordinary unital functional

$\phi : C \to \mathbb{C}$, where C is an arbitrary unital algebra. To put this into our frame we extend C to an algebra over B, namely $C_B := C \otimes B$, and ϕ to the B-functional $\phi_B := \phi \otimes \mathrm{id}$, i.e.

$$\begin{aligned} \phi_B : C_B = C \otimes B &\to B \\ c \otimes b &\mapsto \phi(c)b. \end{aligned}$$

Then we can form $\varphi * \phi_B$ as usual and call this the amalgamated free product of φ and ϕ, also denoted by $\varphi * \phi$.

3.4. Explicit formula for $\varphi_1 * \varphi_2$

In this section, we want to present a formula which allows us to express the moments of $\varphi := \varphi_1 * \varphi_2$ directly in moments of φ_1 and moments of φ_2. In principle, we know how to do this, namely we have to calculate the cumulant functions of φ_1 and φ_2, add them as direct sum to get the cumulant function of φ and calculate from this the moment function of φ. This shows that there exists a formula describing the relation between the moments of φ and those of φ_1 and φ_2, and one may wonder whether it should not be possible to write down this connection more explicitly without having to calculate cumulants.

The first observation in this direction is that one only needs the cumulants of one of the two φ_i, because one can make partial summations in the following way: Let us consider $\varphi(a_1 \dots a_n)$ with $a_1, \dots, a_n \in \mathring{A} = A_1 \cup A_2$. Now we choose a decomposition of $S = (1, \dots, n)$ into $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$, such that $a_i \in A_1$ for $i \in S_1$ and $a_i \in A_2$ for $i \in S_2$. In general (i.e. if some of the a_i are in $A_1 \cap A_2 = B$), there is no unique such decomposition, but the crucial conclusion of all our foregoing considerations on the free product was the insight that the result of our calculations of $\varphi_1 * \varphi_2(a_1 \dots a_n)$ is independent of the special chosen decomposition. Of course, it suffices to consider $a_1 \dots a_n$ where the a_i come alternatingly from A_1 and A_2. But our considerations are not restricted to this case, so we will treat arbitrary $a_1 \dots a_n$ with $a_i \in \mathring{A}$ and fix some decomposition $S = S_1 \cup S_2$. Then, by the definition of $\hat{c} = \hat{c}_1 \oplus \hat{c}_2$, in our formula

$$\varphi(a_1 \dots a_n) = \varphi^{(n)}(a_1 \otimes \dots \otimes a_n) = \sum_{\pi \in NC(n)} \hat{c}(\pi)[a_1 \otimes \dots \otimes a_n]$$

only such $\pi \in NC(n) = NC(S)$ contribute where π does not connect elements from A_1 with elements from A_2, i.e. π must be of the form $\pi = \pi_1 \cup \pi_2$, where $\pi_1 \in NC(S_1)$ and $\pi_2 \in NC(S_2)$. Hence we can split our sum into (comp. our notation in Prop. 1.1.4)

$$\varphi(a_1 \dots a_n) = \sum_{\pi_1 \in NC(S_1)} \sum_{\pi_2 \in NC(\pi_1, S_2)} \hat{c}(\pi_1 \cup \pi_2)[a_1 \otimes \dots \otimes a_n].$$

But now the second sum can be carried out and we obtain with our Notation 2.1.4

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \sum_{\pi_1 \in NC(S_1)} \sum_{\substack{\pi_2 \in NC(S_2) \\ \pi_2 \leq \pi_1^c}} (\hat{c}_1 \cup \hat{c}_2)(\pi_1 \cup \pi_2)[a_1 \otimes \dots \otimes a_n] \\ &= \sum_{\pi_1 \in NC(S_1)} (\hat{c}_1 \cup \hat{\varphi}_2)(\pi_1 \cup \pi_1^c)[a_1 \otimes \dots \otimes a_n]. \end{aligned}$$

In this formula we only need the cumulants of φ_1, but not those of φ_2.

3.4.1. EXAMPLE. Consider again $\varphi(a_1a_2a_3a_4)$ with $a_1, a_3 \in A_1$ and $a_2, a_4 \in A_2$. We have $S_1 = (1,3)$ and the above formula says

$$\begin{aligned}\varphi(a_1a_2a_3a_4) &= (\hat{c}_1 \cup \hat{\varphi}_2)\big(\{(1,3)\} \cup \{(2),(4)\}\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \\ &\quad + (\hat{c}_1 \cup \hat{\varphi}_2)\big(\{(1),(3)\} \cup \{(2,4)\}\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \\ &= c_1^{(2)}\big(a_1\varphi_2(a_2) \otimes a_3\big) \cdot \varphi_2(a_4) + c_1^{(1)}(a_1) \cdot \varphi_2\big(a_2 c_1^{(1)}(a_3) a_4\big),\end{aligned}$$

which reproduces our result from Example 3.3.6.

To get also rid of the cumulants of φ_1 we need the following trick: Let U be a unital $*$-algebra generated by a unitary u, i.e. $uu^* = u^*u = 1$, and equipped with a linear functional $\phi : U \to \mathbb{C}$ with the properties $\phi(1) = 1$ and $\phi(u) = \phi(u^*) = 0$. We extend this to $\phi_B := \phi \otimes \mathrm{id}$ on $U_B := U \otimes B$ and instead of calculating $\varphi_1 * \varphi_2(a_1 \dots a_n)$ (for $a_1, \dots, a_n \in \mathring{A} = A_1 \cup A_2$) in $(A_1 *_B A_2, \varphi_1 * \varphi_2)$ we can of course do this in $(U_B *_B A_1 *_B A_2, \phi_B * \varphi_1 * \varphi_2)$. We will denote in the following $\phi_B * \varphi_1 * \varphi_2$ just by $\phi * \varphi_1 * \varphi_2$. Let as before $(1, \dots, n) = S = S_1 \cup S_2$ be a fixed decomposition such that $a_i \in A_k$ if $i \in S_k$ $(k = 1,2)$. Now we observe that with $(i = 1, \dots, n)$

$$\bar{a}_i := \begin{cases} a_i, & \text{for } i \in S_1 \\ ua_iu^*, & \text{for } i \in S_2 \end{cases}$$

the elements $a_1 \dots a_n$ and $\bar{a}_1 \dots \bar{a}_n$ have the same distribution, i.e.

$$\varphi_1 * \varphi_2(a_1 \dots a_n) = \phi * \varphi_1 * \varphi_2(a_1 \dots a_n) = \phi * \varphi_1 * \varphi_2(\bar{a}_1 \dots \bar{a}_n).$$

This is true, because the moment functions of φ_2 on A_2 and of $\phi * \varphi_2$ on uA_2u^* coincide: For $c_1, \dots, c_k \in A_2$ we have

$$\begin{aligned}(\phi * \varphi_2)^{(k)}(\bar{c}_1 \otimes \dots \otimes \bar{c}_k) &= \phi * \varphi_2(\bar{c}_1 \dots \bar{c}_k) \\ &= \phi * \varphi_2(uc_1 \dots c_ku^*) \\ &= \phi(uu^*)\varphi_2(c_1 \dots c_k) \\ &= \varphi_2(c_1 \dots c_k) \\ &= \varphi_2^{(k)}(c_1 \otimes \dots \otimes c_k).\end{aligned}$$

Thus, instead of $\varphi_1 * \varphi_2(a_1 \dots a_n)$ we can calculate $\phi * \varphi_1 * \varphi_2(\bar{a}_1 \dots \bar{a}_n) = \phi * (\varphi_1 * \varphi_2)(\bar{a}_1 \dots \bar{a}_n)$. If we consider $\bar{a}_1 \dots \bar{a}_n$ as a monomial in a_i, u, u^*, let's say of length $k = n + 2|S_2|$, and decompose $(1, \dots, k) = T_1 \cup T_2$, where T_1 contains the positions of the u and u^* and $T_2 \cong S_1 \cup S_2$ contains the positions of the a_i, then, by our foregoing considerations (for notational convenience, we identify $\varphi_1 * \varphi_2$ with its moment function),

$$\begin{aligned}\varphi_1 * \varphi_2(a_1 \dots a_2) &= \phi * (\varphi_1 * \varphi_2)(\bar{a}_1 \dots \bar{a}_n) \\ &= \sum_{\pi_1 \in NC(T_1)} \big(\hat{c} \cup (\varphi_1 * \varphi_2)\big)(\pi_1 \cup \pi_1^c)[\bar{a}_1 \otimes \dots \otimes \bar{a}_n],\end{aligned}$$

where $\hat{c}$ is the cumulant function of ϕ_B and where an element $\bar{a}_i = ua_iu^*$ (for $i \in S_2$) in the tensor $\bar{a}_1 \otimes \dots \otimes \bar{a}_n$ has to be read as $u \otimes a_i \otimes u^*$, i.e. we are working in $NC(k)$ and not in $NC(n)$. Since $\hat{c}$ takes its values in $\mathbb{C}$, it commutes with everything and, with the canonical identifications $\pi_1^c \in NC(T_2) \cong NC(S)$ and $\pi_1 \in NC(T_1) \cong NC(S_{(2|S_2|)})$, we get

$$\varphi_1 * \varphi_2(a_1 \dots a_n) = \sum_{\pi_1 \in NC(T_1)} \hat{c}(\pi_1)[u \otimes u^* \otimes u \otimes \dots \otimes u^*] \cdot \varphi_1 * \varphi_2(\pi_1^c)[a_1 \otimes \dots \otimes a_n].$$

The crucial observation is now that $\hat{c}(\pi_1)$ vanishes in many cases and the remaining π_1 are of such a form that π_1^c does not couple elements from S_1 with elements from S_2, i.e. $\pi_1^c \in NC(S_1, S_2)$ and hence $\varphi_1 * \varphi_2(\pi_1^c)[a_1 \otimes \dots \otimes a_n]$ decomposes always into a product of moments of φ_1 and moments of φ_2.

3.4.2. PROPOSITION. *Let U be a unital $*$-algebra, $u \in U$ a unitary element and $\phi : U \to \mathbb{C}$ a linear functional with $\phi(1) = 1$. If $\phi(u) = \phi(u^*) = 0$, then the cumulant function $\hat{c} = (c^{(n)}) \in \mathbf{I}^c(U, \mathbb{C})$ of ϕ has the following properties:*
1) For $n = 2k+1$ odd, $c^{(n)}$ vanishes on alternating tensors in u and u^*,

$$c^{(2k+1)}(u \otimes u^* \otimes u \otimes \dots \otimes u) = c^{(2k+1)}(u^* \otimes u \otimes u^* \otimes \dots \otimes u^*) = 0.$$

2) For $n = 2k$ even, we have on alternating tensors

$$c^{(2k)}(u \otimes u^* \otimes u \otimes \dots \otimes u^*) = c^{(2k)}(u^* \otimes u \otimes u^* \otimes \dots \otimes u) = (-1)^{k-1} c_{k-1},$$

where c_k is the Catalan number

$$c_k := \frac{1}{k} \binom{2k}{k-1}.$$

3) The expression $\hat{c}(\pi)[u \otimes u^* \otimes u \otimes \dots \otimes u^*]$ for alternating tensors is only different from zero if $\pi = \{V_1, \dots, V_p\} \in NC(T_1)$ has the property that all $|V_i|$ are even and each V_i $(i = 1, \dots, p)$ connects alternatingly u and u^*. In this case

$$\hat{c}(\pi)[u \otimes u^* \otimes u \otimes \dots \otimes u^*] = (-1)^{|S_2| - p} c_{|V_1|/2-1} \dots c_{|V_p|/2-1}.$$

PROOF. 1) The proof will be by induction on k. For $k = 0$, we have

$$c^{(1)}(u) = \phi(u) = 0 = \phi(u^*) = c^{(1)}(u^*).$$

Now consider $k > 0$ and assume the assertion to be true for all $k' < k$. By the Def. 3.2.1 of a cumulant function and by Lemma 3.2.4, we have with $n = 2k+1$

$$\begin{aligned} c^{(n)}(u \otimes u^* \otimes u \otimes \dots \otimes u) &= \\ &= c^{(n-1)}(1 \otimes u \otimes \dots \otimes u) - \sum_{\substack{\pi \in NC(n) \\ |\pi| = 2 \\ 1 \not\sim_\pi 2}} \hat{c}(\pi)[u \otimes u^* \otimes u \otimes \dots \otimes u] \\ &= - \sum_{\substack{\pi \in NC(n) \\ |\pi| = 2 \\ 1 \not\sim_\pi 2}} \hat{c}(\pi)[u \otimes u^* \otimes u \otimes \dots \otimes u]. \end{aligned}$$

Now the π's in our sum must be of the form

$$\pi = \{(1, l, l+1, \dots, n), (2, 3, \dots, l-1)\} \qquad \text{for some } l \in \{3, \dots, n+1\}.$$

But then $\hat{c}(\pi)[u \otimes u^* \otimes u \otimes \cdots \otimes u]$ vanishes by induction hypothesis, since exactly one of the two blocks of π gives an alternating tensor in u and u^* of odd length $2k' + 1 < 2k + 1$.
For tensors of the form $u^* \otimes u \otimes u^* \otimes \cdots \otimes u^*$ the proof is the same.
2) The proof is again by induction on k. For $k = 1$, we have

$$c^{(2)}(u \otimes u^*) = \phi(uu^*) - \phi(u) \cdot \phi(u^*) = 1 = c_0,$$

and the same for $c^{(2)}(u^* \otimes u)$.
Now consider $k > 1$ and assume the assertion to be true for all $k' < k$. As before we have with $n = 2k$

$$\begin{aligned}
c^{(n)}&(u \otimes u^* \otimes u \otimes \cdots \otimes u^*) = \\
&= -\sum_{l=3}^{n+1} \hat{c}\big((1, l, l+1, \dots, n), (2, 3, \dots, l-1)\big)[u \otimes u^* \otimes u \otimes \cdots \otimes u^*] \\
&= -\sum_{\substack{l=3 \\ l=2k' \text{ even}}}^{n+1} c^{(n-l+2)}(u \otimes u^* \otimes u \otimes \cdots \otimes u^*) c^{(l-2)}(u^* \otimes u \otimes u^* \otimes \cdots \otimes u) \\
&= -\sum_{k'=2}^{k} c^{(2(k-k'+1))}(u \otimes u^* \otimes u \otimes \cdots \otimes u^*) c^{(2(k'-1))}(u^* \otimes u \otimes u^* \otimes \cdots \otimes u) \\
&= -\sum_{k'=2}^{k} (-1)^{k-k'} c_{k-k'} (-1)^{k'-2} c_{k'-2} \\
&= (-1)^{k-1} c_{k-1},
\end{aligned}$$

where only even l survive in the sum, because otherwise $c^{(l-2)}(u^* \otimes u \otimes u^* \otimes \cdots \otimes u^*) = 0$ bei part 1).
The proof for tensors of the form $u^* \otimes u \otimes u^* \otimes \cdots \otimes u$ is the same.
3) This follows directly from 1) and 2). □

3.4.3. Remark. If ϕ fulfills the stronger requirements

$$\phi(u^k) = \phi(u^{*k}) = 0 \qquad \text{for all } k \geq 1,$$

then one can show in the same way that, for $n = 2k + 1$ odd, $c^{(n)}$ vanishes on all tensors in u and u^*,

$$c^{(2k+1)}(u_1 \otimes \cdots \otimes u_{2k+1}) = 0 \qquad \text{for all } u_1, \dots, u_{2k+1} \in \{u, u^*\},$$

and, for $n = 2k$ even, it is only different from zero on the alternating ones $(u_1, \dots, u_{2k} \in \{u, u^*\})$

$$c^{(2k)}(u_1 \otimes \cdots \otimes u_{2k}) = 0 \qquad \text{if } u_i = u_{i+1} \text{ for a } i \in \{1, \dots, 2k-1\}.$$

The value on the alternating tensors is, of course, the one given in part 2) of the foregoing Prop. 3.4.2.

3.4.4. PROPOSITION. *Let as above* $T = T_1 \cup T_2 \cong T_1 \cup S_1 \cup S_2$.
1) If $\pi_1 \in NC(T_1)$ *has the property*

$$\hat{c}(\pi_1)[u \otimes u^* \otimes u \otimes \cdots \otimes u^*] \neq 0,$$

then $\pi_1^c \in NC(T_2) \cong NC(S_1 \cup S_2)$ *does not couple* S_1 *with* S_2*, i.e.*

$$\pi_1^c \in NC(S_1, S_2).$$

2) Conversely, if $\pi \in NC(S_1, S_2) \subset NC(T_2)$*, then we have for* $\pi^c \in NC(T_1)$

$$\hat{c}(\pi^c)[u \otimes u^* \otimes u \otimes \cdots \otimes u^*] \neq 0.$$

PROOF. 1) Assume there exist $i_1 \in S_1$ and $i_2 \in S_2$ with $i_1 \sim_{\pi_1^c} i_2$. Then the interval $]i_1, i_2[\cap T_1 \subset T$ is not empty, because it contains at least the position of the u of $\bar{a}_{i_2} = u a_{i_2} u^*$. Since $]i_1, i_2[\cap T_1$ contains an odd number of elements (all $i \in S_2$ with $i_1 < i < i_2$ contribute both a u and a u^*, only i_2 has an odd contribution) and since each block of π_1 is either disjoint or totally contained in $]i_1, i_2[\cap T_1$, it follows that at least one block of π_1 has an odd number of elements. But then part 3) of Prop. 3.4.2 yields a contradiction to our assumption.
2) By Prop 3.4.2, we have to convince ourselves that all blocks of π^c have an even number of elements and connect alternatingly u with u^*.
Assume that we have $j_1 \sim_{\pi^c} j_2$ where $j_1 < j_2$ are the positions of two u's. We will show that there is a j corresponding to a u^* with $j_1 \sim_{\pi^c} j \sim_{\pi^c} j_2$ and $j_1 < j < j_2$. The successor of j_1, namely $j_1 + 1 \in T$, has to be in S_2. If $V \in \pi$ is the block of π containing this $j_1 + 1$, then $V = (j_1 + 1, \dots, k)$ with $j_1 < j_1 + 1 \leq k < j_2$. Since also k has to be in S_2, the next position, $j := k + 1$, corresponds to a u^*. It is clear that $j_1 \sim_{\pi^c} j$.
Now we show the same for two u^*'s. Let $j_1 \sim_{\pi^c} j_2$ be their positions. The predecessor $j_2 - 1$ belongs to S_2. The block $V \in \pi$ containing this $j_2 - 1$ has to be of the form $V = (k, \dots, j_2 - 1)$ with $j_1 < k \leq j_2 - 1 < j_2$. Now the predecessor of k, $j := k - 1$, must be the position of a u and again we have $j_1 \sim_{\pi^c} j \sim_{\pi^c} j_2$.
It remains to show that all blocks of π^c have an even number of elements. Let us assume that we have a block $V = (k, \dots, l) \in \pi^c$ which starts and ends with the position of a u. Then $l + 1 \in S_2$ belongs to some block $W \in \pi$. If $l + 1$ is not the first element in W, i.e. $W = (\dots, i, l + 1, \dots)$, then $i < k$ and, since $i \in S_2$, we have that $i + 1$ corresponds to a u^*, hence $i + 1 < k$ and $i + 1 \sim_{\pi^c} k$, which yields a contradiction. If $W = (l + 1, \dots, j)$, then $j + 1$ is the position of a u^* and $l \sim_{\pi^c} j + 1$ which is again a contradiction. Thus it is impossible that a block of π^c starts and ends with the position of u's, which implies that the same cannot happen with u^*'s. This proves our assertion. $\square$

Thus the formula

$$\varphi_1 * \varphi_1(a_1 \dots a_n) = \sum_{\pi_1 \in NC(T_1)} \varphi_1 * \varphi_2(\pi_1^c)[a_1 \otimes \cdots \otimes a_n] \cdot \hat{c}(\pi_1)[u \otimes u^* \otimes u \otimes \cdots \otimes u^*]$$

gives us the wanted prescription for expressing moments of $\varphi_1 * \varphi_2$ in terms of moments of φ_1 and moments of φ_2. The rule is to replace in $a_1 \dots a_n$ each a_i for $i \in S_2$ by $u a_i u^*$ and to sum over all $\pi_1 \in NC(T_1)$ which connect alternatingly

u with u^*. The corresponding π_1^c connects then only within A_1 and within A_2, thus $\varphi_1 * \varphi_2(\pi_1^c)[a_1 \otimes \cdots \otimes a_n]$ factorizes into moments of φ_1 and moments of φ_2. The expression $\hat{c}(\pi_1)[u \otimes u^* \otimes u \otimes \cdots \otimes u^*]$ gives a weighting factor for such a contribution. As we said before this is true for all decompositions $S = S_1 \cup S_2$, not only for alternating ones. But in a general decomposition, a $\pi_2 \in NC(T_2)$ may appear several times as $\pi_2 = \pi_1^c$ for different π_1, so in this general case our formula is not a reduced one and there may be cancellations of terms. For example, if we consider $a_1a_2a_3$ for $a_1 \in A_1$ and $a_2, a_3 \in A_2$, then our prescription yields (where all partitions are now considered as elements in $NC(7)$)

$$\begin{aligned}
\varphi_1 * \varphi_2(a_1a_2a_3) &= \phi * \varphi_1 * \varphi_2(a_1ua_2u^*ua_3u^*) \\
&= \varphi_1 * \varphi_2\big((1),(3),(6)\big)[a_1 \otimes a_2 \otimes a_3] \cdot \hat{c}\big((2,4,5,7)\big)[u \otimes u^* \otimes u \otimes u^*] \\
&\quad + \varphi_1 * \varphi_2\big((1),(3),(6)\big)[a_1 \otimes a_2 \otimes a_3] \cdot \hat{c}\big((2,4),(5,7)\big)[u \otimes u^* \otimes u \otimes u^*] \\
&\quad + \varphi_1 * \varphi_2\big((1),(3,6)\big)[a_1 \otimes a_2 \otimes a_3] \cdot \hat{c}\big((2,7),(4,5)\big)[u \otimes u^* \otimes u \otimes u^*] \\
&= -\varphi_1(a_1) \cdot \varphi_2(a_2) \cdot \varphi_2(a_3) \\
&\quad + \varphi_1(a_1) \cdot \varphi_2(a_2) \cdot \varphi_2(a_3) \\
&\quad + \varphi_1(a_1) \cdot \varphi_2(a_2a_3) \\
&= \varphi_1(a_1) \cdot \varphi_2(a_2a_3),
\end{aligned}$$

since

$$c^{(2)}(u \otimes u^*) = c^{(2)}(u^* \otimes u) = 1 \qquad \text{and} \qquad c^{(4)}(u \otimes u^* \otimes u \otimes u^*) = -1.$$

This result follows, of course, much simpler, if we write $a_1a_2a_3$ in the form $a_1(a_2a_3)$ with $a_1 \in A_1$, $a_2a_3 \in A_2$. Then we have (now in $NC(4)$)

$$\begin{aligned}
\varphi_1 * \varphi_2(a_1a_2a_3) &= \phi * \varphi_1 * \varphi_2\big(a_1u(a_2a_3)u^*\big) \\
&= \varphi_1 * \varphi_2\big((1),(3)\big)[a_1 \otimes a_2a_3] \cdot \hat{c}\big((2,4)\big)[u \otimes u^*] \\
&= \varphi_1(a_1) \cdot \varphi_2(a_2a_3).
\end{aligned}$$

Of course, we can always restrict to the consideration of decompositions $S = S_1 \cup S_2$ where S_2 is separated by S_1. This implies that T_1 is separated by T_2 and then, by Prop. 1.1.4, we have for all $\pi_1 \in NC(T_1)$ that $(\pi_1^c)^c = \pi_1$. Hence each $\pi_2 \in NC(T_2)$ appears at most once in our summation (namely, if $\pi_2 = \pi_1^c$ for a $\pi_1 \in NC(T_1)$), there cannot occur cancellations and our formula is in a reduced form. If $S = S_1 \cup S_2$ is even an alternating decomposition, then $T = T_1 \cup T_2$ is alternating, too, and each $\pi_2 \in NC(T_2)$ is of the form $\pi_2 = \pi_1^c$ (for $\pi_1 = \pi_2^c$) and we can rewrite our formula in the form

$$\begin{aligned}
&\varphi_1 * \varphi_2(a_1 \dots a_n) = \\
&= \sum_{\pi_2 \in NC(S_1,S_2) \subset NC(T_1)} \varphi_1 * \varphi_2(\pi_2)[a_1 \otimes \cdots \otimes a_n] \cdot \hat{c}(\pi_2^c)[u \otimes u^* \otimes u \otimes \cdots \otimes u^*].
\end{aligned}$$

In this case, all possible non-crossing factorizations of $a_1 \dots a_n$ into blocks in A_1 and blocks in A_2 give a non-vanishing contribution.

3.4.5. EXAMPLES. 1) Consider again $a_1a_2a_3$ with $a_1, a_3 \in A_1$ and $a_2 \in A_2$. Then $S = (1,2,3) = (1,3) \cup (2)$ and S_2 is separated by S_1. We have in $NC(5)$

$$\begin{aligned}\varphi_1 * \varphi_2(a_1a_2a_3) &= \phi * \varphi_1 * \varphi_2(a_1ua_2u^*a_3)\\ &= \varphi_1 * \varphi_2\big((1,5),(3)\big)[a_1 \otimes a_2 \otimes a_3] \cdot \hat{c}\big((2,4)\big)[u \otimes u^*]\\ &= \varphi_1\big(a_1\varphi_2(a_2)a_3\big).\end{aligned}$$

Note that the decomposition of S is not alternating, because $(1,3) \subset S_1$ is not separated by a pair in S_2. This corresponds to the fact that the term $\varphi_1(a_1)\varphi_2(a_2)\varphi_1(a_3)$ does not appear.
2) Consider $a_1a_2a_3a_4$ with $a_1, a_3 \in A_1$ and $a_2, a_4 \in A_2$, compare Example 3.3.6. Then the decomposition $S = (1,2,3,4) = (1,3) \cup (2,4)$ is alternating and all possible factorizations corresponding to $NC\big((1,3),(2,4)\big)$ appear:

$$\begin{aligned}&\varphi_1 * \varphi_2(a_1a_2a_3a_4) = \phi * \varphi_1 * \varphi_2(a_1ua_2u^*a_3ua_4u^*)\\ &= \varphi_1 * \varphi_2\big((1),(3),(5),(7)\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \cdot \hat{c}\big((2,4,6,8)\big)[u \otimes u^* \otimes u \otimes u^*]\\ &\quad + \varphi_1 * \varphi_2\big((1,5),(3),(7)\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \cdot \hat{c}\big((2,4),(6,8)\big)[u \otimes u^* \otimes u \otimes u^*]\\ &\quad + \varphi_1 * \varphi_2\big((1),(3,7),(5)\big)[a_1 \otimes a_2 \otimes a_3 \otimes a_4] \cdot \hat{c}\big((2,8),(4,6)\big)[u \otimes u^* \otimes u \otimes u^*]\\ &= -\varphi_1(a_1) \cdot \varphi_2(a_2) \cdot \varphi_1(a_3) \cdot \varphi_2(a_4)\\ &\quad + \varphi_1\big(a_1\varphi_2(a_2)a_3\big) \cdot \varphi_2(a_4)\\ &\quad + \varphi_1(a_1) \cdot \varphi_2\big(a_2\varphi_1(a_3)a_4\big),\end{aligned}$$

yielding directly the reduced result of Example 3.3.6.

3.5. Positivity of the amalgamated free product

Up to now we have only talked about B-functionals without equipping them with some positivity property. Here, we want to treat such questions, in particular, we ask about the positivity of our free product construction, i.e., whether $*\varphi_i$ is positive, if the φ_i are. For the notion of positivity, we need of course a $*$-structure on our algebras A and B. It will also be essential that B has a nice positivity structure, therefore we demand it to be a C^*-algebra. For the $*$-algebra A no such restriction is required.

Since our functionals are in general not scalar-valued but take their values in the C^*-algebra B, one has to distinguish, a priori, between the notions of 'positive' and 'completely positive'. But for conditional expectations, the only case we consider, these two notions coincide.

For the C^*-algebra B each positive element is of the form bb^* for some $b \in B$. Correspondingly, we will also call an element of the form aa^* $(a \in A)$ for our $*$-algebra A a positive element.

By $M_n(A) = M_n \otimes A$, we denote the algebra of all $n \times n$-matrices with entries from A, equipped with the canonical $*$-structure: For $\alpha = (a_{ij})_{i,j=1}^n \in M_n(A)$, we have

$$\alpha^* = (\bar{a}_{ij})_{i,j=1}^n \qquad \text{with} \qquad \bar{a}_{ij} = a_{ji}^*.$$

If B is a C^*-algebra, then, of course, $M_n(B)$ is a C^*-algebra, too.

3.5.1. Definition. Let B be a unital C^*-algebra and A a unital $*$-algebra. Consider a unital linear map $\varphi : A \to B$.
1) We say that $\varphi : A \to B$ is *positive* if

$$\varphi(aa^*) \geq 0 \qquad \text{for all } a \in A,$$

i.e. if there exists for each $a \in A$ an element $b \in B$ such that $\varphi(aa^*) = bb^*$.
2) We say that $\varphi : A \to B$ is *completely positive* if, for each $n \in \mathbb{N}$, the map

$$\begin{aligned} \varphi \otimes \mathrm{id}_n : M_n(A) &\to M_n(B) \\ (a_{ij})_{i,j=1}^n &\mapsto \big(\varphi(a_{ij})\big)_{i,j=1}^n \end{aligned}$$

is positive.

The following trivial facts about our notations can, for example, be found in [Pau].

3.5.2. Lemma. *1) Each positive element $\alpha\alpha^*$ in $M_n(A)$ ($\alpha \in M_n(A)$) can be written as*

$$\alpha\alpha^* = \sum_{k=1}^n \big(a_i^{(k)} a_j^{(k)*}\big)_{i,j=1}^n \in M_n(A)$$

for some $a_i^{(k)} \in A$ $(i, k = 1, \dots, n)$.
2) Let $\varphi : A \to B$ be a unital linear map. Then the following statements are equivalent:

a) *The map $\varphi : A \to B$ is completely positive.*
b) *For each $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in A$, the element $\big(\varphi(a_i a_j^*)\big)_{i,j=1}^n \in M_n(B)$ is positive.*
c) *For each $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in A$, there exist elements $b_i^{(k)} \in B$ $(i, k = 1, \dots, n)$ such that we have*

$$\varphi(a_i a_j^*) = \sum_{k=1}^n b_i^{(k)} b_j^{(k)*} \qquad \textit{for all } i, j = 1, \dots, n.$$

Proof. 1) Write $\alpha = (a_{ij})_{i,j=1}^n$. Then

$$\alpha\alpha^* = \sum_{k=1}^n \big(a_i^{(k)} a_j^{(k)*}\big)_{i,j=1}^n \qquad \text{with} \qquad a_i^{(k)} := a_{ik}.$$

2) a) $\Rightarrow$ b): Since $(a_i a_j^*)_{i,j=1}^n$ is positive in $M_n(A)$, this implication follows by the very definition of 'completely positive'.
b) $\Rightarrow$ c): Consider the matrix $\big(\varphi(a_i a_j^*)\big)_{i,j=1}^n \in M_n(B)$. By b), it is positive and hence part 1) ensures that it can be written in the form

$$\big(\varphi(a_i a_j^*)\big)_{i,j=1}^n = \sum_{k=1}^n \big(b_i^{(k)} b_j^{(k)*}\big)_{i,j=1}^n$$

for some $b_i^{(k)} \in B$ $(i, k = 1, \ldots, n)$. Comparing the entry (i, j) on both sides gives statement c).
c) $\Rightarrow$ a): We have to show that $\varphi \otimes \mathrm{id}_n(\alpha\alpha^*) \geq 0$ in $M_n(B)$ for all $n \in \mathbb{N}$ and all $\alpha \in M_n(A)$. By 1), we have

$$\alpha\alpha^* = \sum_{k=1}^{n} \big(a_i^{(k)} a_j^{(k)*}\big)_{i,j=1}^{n}$$

for some $a_i^{(k)} \in A$. Thus c) gives

$$\begin{aligned} \varphi \otimes \mathrm{id}_n(\alpha\alpha^*) &= \sum_{k=1}^{n} \big(\varphi(a_i^{(k)} a_j^{(k)*}\big)_{i,j=1}^{n} \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} \big(b_i^{(k,l)} b_j^{(k,l)*}\big)_{i,j=1}^{n} \\ &\geq 0. \quad \square \end{aligned}$$

The coincidence of 'positivity' and 'complete positivity' for B-functionals relies essentially on the following characterization of positive elements in $M_n(B)$. For a proof we refer to [Pas].

3.5.3. LEMMA. *Let B be a unital C^*-algebra. Let $n \in \mathbb{N}$ and $\tilde{b}_{ij} \in B$ $(i, j = 1, \ldots, n)$. Then the following statements are equivalent.*

a) *The matrix $\big(\tilde{b}_{ij}\big)_{i,j=1}^{n} \in M_n(B)$ is positive.*

b) *We have*

$$\sum_{i,j=1}^{n} b_i \tilde{b}_{ij} b_j^* \geq 0 \qquad \textit{for all } b_1, \ldots, b_n \in B.$$

The preceding two lemmas have now the following implications for positive B-functionals.

3.5.4. PROPOSITION. *Let B be a unital C^*-algebra, A a $*$-algebra over B, and $\varphi : A \to B$ a positive B-functional. Then φ has the following properties.*
1) φ is completely positive.
2) For each $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in A$ there exist elements $b_i^{(k)} \in B$ $(i, k = 1, \ldots, n)$ such that we have

$$\varphi(a_i a_j^*) = \sum_{k=1}^{n} b_i^{(k)} b_j^{(k)*} \qquad \textit{for all } i, j = 1, \ldots, n.$$

PROOF. By 3.5.2, it suffices to show that for each $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in A$ the matrix $\big(\varphi(a_i a_j^*)\big)_{i,j=1}^{n} \in M_n(B)$ is positive. But now we have for all $b_1, \ldots, b_n \in B$ that

$$\sum_{i,j=1}^{n} b_i \varphi(a_i a_j^*) b_j^* = \sum_{i,j=1}^{n} \varphi(b_i a_i a_j^* b_j^*) = \varphi\Big(\big(\sum_{i=1}^{n} b_i a_i\big)\big(\sum_{j=1}^{n} b_j a_j\big)^*\Big) \geq 0,$$

which implies, by 3.5.3, the assertion. □

3.5.5. Remark. Note that positivity of a B-functional φ implies in particular $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

Now we are in the position to prove our main result concerning the positivity of the amalgamated free product.

3.5.6. Theorem. *Let B be a unital C^*-algebra and let $\varphi_i : A_i \to B$ $(i \in I)$ be positive B-functionals. Then the amalgamated free product*

$$\varphi := *_{i\in I}\varphi_i : \underset{i\in I}{*_B} A_i \to B$$

is positive, too.

Proof. We have to show

$$\varphi(aa^*) \geq 0 \qquad \text{for all } a \in \underset{i\in I}{*_B} A_i.$$

We can write each such a as a finite sum of elements of the form $a_1 \dots a_n$, where $n \in \mathbb{N}_0$, $a_k \in A_{i(k)}$, $i(k) \neq i(k+1)$ for $k = 1, \dots, n-1$, and $\varphi(a_k) = 0$ for $k = 1, \dots, n$. (For $n = 0$, this shall mean that a is a multiple of the unit 1.) If we have two such elements, say $a_1 \dots a_n$ and $\tilde{a}_1 \dots \tilde{a}_m$, then, by Prop. 3.3.3, we have

$$\varphi(a_1 \dots a_{n-1} a_n \tilde{a}_m \tilde{a}_{m-1} \dots \tilde{a}_1) = \delta_{nm}\varphi\Big(a_1 \dots \varphi\big(a_{n-1}\varphi(a_n\tilde{a}_n)\tilde{a}_{n-1}\big) \dots \tilde{a}_1\Big).$$

This implies in particular that elements of this form with different length n or different tuples $(i(1), \dots, i(n))$ are orthogonal with respect to φ, hence it suffices to consider elements a of the form $a = \sum_{k=1}^r a_1^{(k)} \dots a_n^{(k)}$, where $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, $a_j^{(k)} \in A_{i(j)}$ (for all $k = 1, \dots, r$) with $i(j) \neq i(j+1)$ for $j = 1, \dots, n-1$ and $\varphi(a_j^{(k)}) = 0$ for all $j = 1, \dots, n$ and $k = 1, \dots, r$. Then we have

$$\varphi(aa^*) = \sum_{k,l=1}^{r} \varphi\Big(a_1^{(k)} \dots \varphi\big(a_{n-1}^{(k)}\varphi(a_n^{(k)}a_n^{(l)*})a_{n-1}^{(l)*}\big) \dots a_1^{(l)*}\Big).$$

Since $\varphi_{i(n)}$ is positive, Prop. 3.5.4 tells us that we can write

$$\varphi(a_n^{(k)}a_n^{(l)*}) = \varphi_{i(n)}(a_n^{(k)}a_n^{(l)*}) = \sum_{r_n=1}^{r} b_{r_n}^{(k)}b_{r_n}^{(l)*} \qquad \text{for all } k, l = 1, \dots, r$$

with some $b_{r_n}^{(k)} \in B$ $(r_n, k = 1, \dots, r)$. Hence

$$\varphi\big(a_{n-1}^{(k)}\varphi(a_n^{(k)}a_n^{(l)*})a_{n-1}^{(l)*}\big) = \sum_{r_n=1}^{r} \varphi_{i(n-1)}\big(a_{n-1}^{(k)}b_{r_n}^{(k)}b_{r_n}^{(l)^*}a_{n-1}^{(l)*}\big)$$

and we can repeat the same argument as above by replacing $\varphi_{i(n)}$ by $\varphi_{i(n-1)}$ and $a_n^{(k)} \in A_{i(n)}$ by $a_{n-1}^{(k)}b_{r_n}^{(k)} \in A_{i(n-1)}$, yielding

$$\varphi\big(a_{n-1}^{(k)}\varphi(a_n^{(k)}a_n^{(l)*})a_{n-1}^{(l)*}\big) = \sum_{r_{n-1}=1}^{r}\sum_{r_n=1}^{r} b_{r_{n-1},r_n}^{(k)} b_{r_{n-1},r_n}^{(l)*}$$

for some $b^{(k)}_{r_{n-1},r_n} \in B$ $(r_{n-1}, r_n, k = 1, \dots, r)$. We can go on in this way until we end up with

$$\begin{aligned}\varphi(aa^*) &= \sum_{k,l=1}^{r} \sum_{r_1=1}^{r} \cdots \sum_{r_n=1}^{r} b^{(k)}_{r_1,\dots,r_n} b^{(l)*}_{r_1,\dots,r_n} \\ &= \sum_{r_1=1}^{r} \cdots \sum_{r_n=1}^{r} \Big(\sum_{k=1}^{r} b^{(k)}_{r_1,\dots,r_n}\Big)\Big(\sum_{l=1}^{r} b^{(l)}_{r_1,\dots,r_n}\Big)^* \\ &\geq 0. \quad \square\end{aligned}$$