CHAPTER I

PRELIMINARIES ON NON-CROSSING PARTITIONS

In this first chapter, we want to collect all relevant combinatorial facts about the lattice of non-crossing partitions. In particular, we will examine the notion of (scalar-valued) multiplicative functions on this lattice, a notion which will be generalized in the next chapter to the crucial concept of operator-valued multiplicative functions. The material presented here is essentially taken from [Spe4] and deeply inspired by Rota's [Rot1,Rot2] combinatorial point of view on classical probability theory.

1.1. Non-crossing partitions

We start by recalling the basic definitions and facts about non-crossing partitions.

1.1.1. Definition. Let S be a linearly ordered set. Then $\pi = \{V_1, \dots, V_p\}$ is a *partition* of S, if the $V_i \neq \emptyset$ are disjoint sets, whose union is S. The partition π is called *non-crossing* if for all $i, j = 1, \dots, p$ with $V_i = \{v_1, \dots, v_n\}$ $(v_1 < \dots < v_n)$ and $V_j = \{w_1, \dots, w_m\}$ $(w_1 < \dots < w_m)$ we have

$$w_k < v_1 < w_{k+1} \Leftrightarrow w_k < v_n < w_{k+1} \qquad (k = 1, \dots, m-1).$$

We can reformulate the definition of 'non-crossing' in a recursive way: The partition $\pi = \{V_1, \dots, V_p\}$ is non-crossing if at least one V_i is an interval in S (i.e. it contains exactly all points of S lying between two points) and $\pi \backslash \{V_i\}$ is a non-crossing partition of $S \backslash V_i$.
We will denote the set of all non-crossing partitions of S by $NC(S)$.

1.1.2. Example. Non-crossing partitions of $\{1, 2, 3, 4, 5\}$ are

$$\big\{\{1,3,5\},\{2\},\{4\}\big\} \qquad \text{and} \qquad \big\{\{1,5\},\{2,4\},\{3\}\big\},$$

crossing ones are

$$\big\{\{1,3\},\{2,4,5\}\big\} \qquad \text{and} \qquad \big\{\{1,4\},\{2\},\{3,5\}\big\}.$$

1.1.3. Notations. 1) The V_i are called the *blocks* of a given partition $\pi = \{V_1, \dots, V_p\}$. If we write a block in the form $V_i = (v_1, \dots, v_m)$, then this will always imply that $v_1 < \dots < v_m$. A $\pi \in NC(S)$ determines an equivalence relation on S in a canonical way and we will write $v \sim_\pi w$, if $v, w \in S$ belong to the same block of π. By $|\pi|$ we will denote the number of blocks of π, and in the same way by $|V|$ the number of elements of the set V. For $s_1, s_2 \in S$ with $s_1 \leq s_2$, we call

$$[s_1, s_2] := \{s \in S \mid s_1 \leq s \leq s_2\} \qquad \text{and} \qquad]s_1, s_2[:= \{s \in S \mid s_1 < s < s_2\}$$

a closed and an open *interval* in S, respectively.
2) One can define a partial order structure on $NC(S)$ exactly as in the case of all partitions: $\pi_1 \leq \pi_2$ if and only if each block of π_1 is contained in one block of π_2. With this order, $NC(S)$ becomes a lattice. Subsets of $NC(S)$ of the form

$$[\pi, \sigma] := \{\nu \in NC(S) \mid \pi \leq \nu \leq \sigma\} \qquad \text{for } \pi, \sigma \in NC(S)$$

are called *segments* of $NC(S)$. Of course, the lattice structure of $NC(S)$ depends only on the number of elements of S, i.e. we can restrict for each n to the consideration of some fixed $S_{(n)}$ with $\#S_{(n)} = n$, say $S_{(n)} := (1, \dots, n)$. In the following we will often freely identify a $\pi \in NC(S)$ for $\#S = n$ with the corresponding element in $NC(S_{(n)})$. By $\mathbf{0}_n := \{(i) \mid i \in S_{(n)}\}$ and $\mathbf{1}_n := \{S_{(n)}\}$ we denote the minimal and the maximal element of $NC(n) := NC(S_{(n)})$, consisting of n blocks and one block, respectively. More generally, we will denote the one block partition of a set S by $\mathbf{1}_S$.
3) For a given $\pi \in NC(n)$, we denote by $\vec{\pi} \in NC(n)$ that partition which results from π by a cyclic permutation of the elements of $S_{(n)}$, i.e. the blocks of $\vec{\pi}$ arise from the blocks of π by adding 1 (mod n) to each element (and then bringing the blocks again into natural linear order). For example, for $\pi = \{(1,2),(3,4,5)\} \in NC(5)$ we get $\vec{\pi} = \{(2,3),(1,4,5)\} \in NC(5)$. It is easy to see that the non-crossing property of π implies the non-crossing property of $\vec{\pi}$. Indeed, often non-crossing partitions are visualized not on the linear ordered line, but on the circle. There the transition from π to $\vec{\pi}$ corresponds to a rotation.
4) If we have a decomposition of S into two disjoint subsets, $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$, then we denote by $NC(S_1, S_2)$ those $\pi \in NC(S)$ which are adapted to this decomposition, i.e.

$$NC(S_1, S_2) := \{\pi \in NC(S_1 \cup S_2) \mid \pi = \pi_1 \cup \pi_2, \pi_1 \in NC(S_1), \pi_2 \in NC(S_2)\}.$$

We say that a pair $(i_1, i_2) \subset S_1$ *separates* a pair $(i_3, i_4) \subset S_2$ if either $i_1 < i_3 < i_2 < i_4$ or $i_3 < i_1 < i_4 < i_2$. We will denote this (symmetric) situation by $(i_1, i_2) \bowtie (i_3, i_4)$. If each pair in S_1 is separated by some pair in S_2, then we say that S_2 separates S_1. Note that 'S_2 separates S_1' and 'S_1 separates S_2' is not the same. We call a decomposition $S = S_1 \cup S_2$ *alternating* if S_1 separates S_2 and S_2 separates S_1. This means, of course, nothing else but that $|S|$ has to be even and that S_1 contains the first, third, fifth, etc. element of S and S_2 contains the second, fourth, sixth, etc. element of S (or vice versa). For example, for $S = S_{(6)}$, we have the alternating decomposition

$$S_{(6)} = (1,2,3,4,5,6) = (1,3,5) \cup (2,4,6).$$

1.1.4. Proposition. *Let a decomposition $S = S_1 \cup S_2$ be given.*
1) Then, for each $\pi_1 \in NC(S_1)$, there exists a maximal element, $\pi_1^c \in NC(S_2)$, of the set

$$NC(\pi_1, S_2) := \{\pi_2 \in NC(S_2) \mid \pi_1 \cup \pi_2 \in NC(S_1, S_2)\},$$

and, for each $\pi_2 \in NC(S_2)$, there exists a maximal element, $\pi_2^c \in NC(S_1)$, of the set

$$NC(S_1, \pi_2) := \{\pi_1 \in NC(S_1) \mid \pi_1 \cup \pi_2 \in NC(S_1, S_2)\}.$$

2) The following statements are equivalent:

a) *We have for all* $\pi_1 \in NC(S_1)$

$$(\pi_1^c)^c = \pi_1.$$

b) *The set* S_2 *separates* S_1*, i.e. for each pair* $(i_1, i_2) \subset S_1$ *there exists a pair* $(i_3, i_4) \subset S_2$ *such that* $(i_1, i_2) \bowtie (i_3, i_4)$*.*

c) *For all* $\pi_1 \in NC(S_1)$ *and all pairs* $(i_1, i_2) \subset S_1$ *with* $i_1 \not\sim_{\pi_1} i_2$ *there exists a pair* $(i_3, i_4) \subset S_2$ *with* $i_3 \sim_{\pi_1^c} i_4$ *such that* $(i_1, i_2) \bowtie (i_3, i_4)$*.*

3) The following statements are equivalent:

a) *We have for all* $\pi_1 \in NC(S_1)$ *and for all* $\pi_2 \in NC(S_2)$

$$(\pi_1^c)^c = \pi_1 \qquad \textit{and} \qquad (\pi_2^c)^c = \pi_2.$$

b) *The decomposition* $S = S_1 \cup S_2$ *is alternating.*

PROOF. 1) We define π_1^c explicitly by the requirement, that $(i_3, i_4) \subset S_2$ are in the same block of π_1^c if there are no $(i_1, i_2) \subset S_1$ with $i_1 \sim_{\pi_1} i_2$ such that $(i_1, i_2) \bowtie (i_3, i_4)$. It is then clear that $\pi_1^c = \max NC(\pi_1, S_2)$. The definition of π_2^c is analogous.
2) a) $\Rightarrow$ b): Assume $(\pi_1^c)^c = \pi_1$ for all $\pi_1 \in NC(S_1)$. If b) were not true, then we could find a pair $(i_1, i_2) \subset S_1$ which is not separated by a pair in S_2. By definition of π_2^c, this implies $i_1 \sim_{\pi_2^c} i_2$ for all $\pi_2 \in NC(S_2)$. If we take now

$$\pi_1 := \{(i_1), (i_2), S_1 \backslash (i_1, i_2)\}, \qquad \pi_2 = \pi_1^c,$$

then $i_1 \not\sim_{\pi_1} i_2$, but $i_1 \sim_{(\pi_1^c)^c} i_2$, thus $\pi_1 \neq (\pi_1^c)^c$.
b) $\Rightarrow$ c): Consider $\pi_1 \in NC(S_1)$ and $(i_1, i_2) \subset S_1$ with $i_1 \not\sim_{\pi_1} i_2$. There are now different alternatives which have to be checked separately, we will only consider the case $i_1 \in V_1$, $i_2 \in V_2$, $V_1, V_2 \in \pi_1$ such that $V_1 = (\dots, i_1, i \dots)$, $V_2 = (k, \dots, i_2, \dots, l)$ with $i_1 < k \leq i_2 \leq l < i$. Then, by b), the successor of l, let's call it i_4, and the predecessor of k, let's call it i_3, must be in S_2 and it is clear that (i_3, i_4) is not separated by some block of π_1, thus $i_3 \sim_{\pi_1^c} i_4$. Hence $(i_3, i_4) \subset S_2$ is the wanted pair with $(i_1, i_2) \bowtie (i_3, i_4)$.
c) $\Rightarrow$ a): Note that our definition of π_2^c implies that for all pairs $(i_1, i_2) \subset S_1$ and $(i_3, i_4) \subset S_2$ with $(i_1, i_2) \bowtie (i_3, i_4)$ we have

$$i_3 \sim_{\pi_2} i_4 \Longrightarrow i_1 \not\sim_{\pi_2^c} i_2.$$

Consider now $\pi_1 \in NC(S_1)$. Since, by definition, $(\pi_1^c)^c \geq \pi_1$, it suffices to show that $i_1 \not\sim_{\pi_1} i_2$ $(i_1, i_2 \in S_1)$ implies $i_1 \not\sim_{(\pi_1^c)^c} i_2$. But by c), for a given pair $(i_1, i_2) \subset S_1$ with $i_1 \not\sim_{\pi_1} i_2$ there exists a pair $(i_3, i_4) \subset S_2$ with $i_3 \sim_{\pi_1^c} i_4$ such that $(i_1, i_2) \bowtie (i_3, i_4)$, which, by the above remark, implies $i_1 \not\sim_{(\pi_1^c)^c} i_2$. This proves a).
3) This follows directly by 2) and the definition of an alternating decomposition. ☐

1.1.5. EXAMPLE. Choose the non-alternating decomposition

$$S_{(14)} = S_1 \cup S_2 = (2, 4, 6, 10, 13) \cup (1, 3, 5, 7, 8, 9, 11, 12, 14).$$

Then S_1 is separated by S_2 and hence $(\pi_1^c)^c = \pi_1$ for all $\pi_1 \in NC(S_1)$, e.g., for

$$\pi_1 = \{(2,13),(4,6,10)\}$$

we have

$$\pi_1^c = \{(1,14),(3,11,12),(5),(7,8,9)\} \quad \text{and} \quad (\pi_1^c)^c = \{(2,13),(4,6,10)\} = \pi_1.$$

On the other hand, S_2 is not separated by S_1, namely the pairs $(7,8)$, $(7,9)$, $(8,9)$, $(11,12)$, and $(1,14)$ cannot be separated by pairs from S_1. So, e.g., for

$$\pi_2 = \{(1),(3,11,14),(5,8),(7),(9),(12)\}$$

we have

$$\pi_2^c = \{(2),(4,10),(6),(13)\} \quad \text{and} \quad (\pi_2^c)^c = \{(1,3,11,12,14),(5,7,8,9)\} > \pi_2.$$

1.1.6. Remarks. 1) The lattice of non-crossing partitions was introduced by Kreweras [Kre] in 1972. Since then there have appeared some purely combinatorial investigations on this lattice, e.g. [Pou,Ede1,Ede2,BSS,ES,Sim,SU,Bia3], but the connection of this lattice with free products and free probability theory, as we will use it here, is quite new and was realized for the first time in [Spe4]. This connection between non-crossing partitions and free probability theory served also as one starting point for Nica's investigations of q-deformations of free convolution [Nic2,Nic3].
2) In [Spe1,Spe2], we used the notion 'admissible' instead of 'non-crossing'.
3) In the case of an alternating decomposition, the map $\pi_1 \mapsto \pi_1^c$ was introduced by Kreweras [Kre], see also the proof of Simion and Ullman [SU] that the lattice $NC(n)$ is self-dual.

1.2. Incidence algebra and convolution

Now we will use the lattice structure for defining a convolution for functions on the lattice of non-crossing partitions (not to be mixed up with the later definition of free convolution). This is a standard procedure for such lattices or, more generally, for partially ordered sets (posets), see [Rot1,Rot2].

1.2.1. Definition. 1) The (large) *incidence algebra* $\mathbf{I}_2$ is defined as the set of all complex-valued functions $\eta(\pi,\sigma)$ on $\bigcup_{n\in\mathbb{N}}\big(NC(n)\times NC(n)\big)$ with the property that $\eta(\pi,\sigma)=0$ if $\pi \not\leq \sigma$. The set $\mathbf{I}_2$ becomes an algebra under the convolution

$$\begin{aligned} \star : \mathbf{I}_2 \times \mathbf{I}_2 &\to \mathbf{I}_2 \\ (\theta,\eta) &\mapsto \theta\star\eta, \end{aligned}$$

where

$$(\theta\star\eta)(\pi,\sigma) := \sum_{\substack{\nu\in NC(n)\\ \pi\leq\nu\leq\sigma}} \theta(\pi,\nu)\eta(\nu,\sigma)$$

for $\pi,\sigma \in NC(n)$.
2) We can also consider functions $\Theta(\pi)$ of one variable, defined on $NC(\infty) :=$

$\bigcup_{n\in\mathbb{N}} NC(n)$. Let us denote the set of these functions by $\mathbf{I}$. We can then also define a convolution between elements of $\mathbf{I}$ and $\mathbf{I}_2$:

$$\begin{aligned}\star : \mathbf{I} \times \mathbf{I}_2 &\to \mathbf{I}\\ (\Theta, \eta) &\mapsto \Theta \star \eta,\end{aligned}$$

where

$$(\Theta \star \eta)(\sigma) := \sum_{\substack{\nu \in NC(n)\\ \nu \leq \sigma}} \Theta(\nu)\eta(\nu, \sigma)$$

for $\sigma \in NC(n)$. Note that this coincides with the previous definition if we consider $\Theta \in \mathbf{I}$ as the restriction of a function $\theta \in \mathbf{I}_2$ via $\Theta(\pi) = \theta(\mathbf{0}_n, \pi)$ for $\pi \in NC(n)$.

1.2.2. EXAMPLES. There are some special functions of prominent interest which deserve their own names, namely the *zeta function*

$$\zeta(\pi, \sigma) = \begin{cases} 1, & \text{if } \pi \leq \sigma \\ 0, & \text{otherwise,} \end{cases}$$

the *delta function* (which is the unit of the incidence algebra $\mathbf{I}_2$)

$$\delta(\pi, \sigma) = \begin{cases} 1, & \text{if } \pi = \sigma \\ 0, & \text{otherwise,} \end{cases}$$

and the *Möbius function* $\mu \in \mathbf{I}_2$, which is defined as the inverse of ζ with respect to the convolution in $\mathbf{I}_2$.

1.3. Multiplicative functions

Whereas up to now we have not used any relation between the $NC(n)$ for different n, we shall now concentrate on the notion of multiplicative functions, which requires that all segments $[\pi, \sigma]$ of $NC(m)$ are isomorphic to the product of some $NC(n)$.

We introduced this notion of a multiplicative function in [Spe4], mainly motivated by the corresponding concept of multiplicative functions on the lattice of all partitions and the observation that multiplicative functions are the right language to talk about the connection between moments and cumulants.

1.3.1. PROPOSITION. *Let $\pi, \sigma \in NC(m)$ with $\pi \leq \sigma$. Then there exists a canonical sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$, such that*

$$[\pi, \sigma] \cong \prod_{n\in\mathbb{N}} NC(n)^{k_n}.$$

Note that the infinite sequence and the infinite product are actually finite ones, since $k_n = 0$ at least for $n > m$. Note also the ambiguity of k_1. Since $NC(1) = \{\mathbf{0}_1 = \mathbf{1}_1\}$, the formula of our proposition can not characterize k_1; nevertheless we shall define it uniquely in the proof of the proposition.

PROOF. Consider $\pi, \sigma \in NC(m)$ with $\pi \leq \sigma$. This means that each block V_i of $\pi = \{V_1, \dots, V_p\}$ is contained in exactly one block W_j of $\sigma = \{W_1, \dots, W_r\}$. Let us denote by $\pi \cap W_j$ the partition of W_j given by

$$\pi \cap W_j := \{V_1 \cap W_j, \dots, V_p \cap W_j\},$$

where the empty set has to be deleted. One sees easily that one has

$$[\pi,\sigma]\cong[\pi\cap W_1,\mathbf{1}_{W_1}]\times\cdots\times[\pi\cap W_r,\mathbf{1}_{W_r}].$$

Thus it is sufficient to prove the assertion for $[\pi,\mathbf{1}_m]$ for all $m\in\mathbb{N}$ and all $\pi\in NC(m)$. We shall now give a recursive procedure for finding the sequence $(k_n)_{n\in\mathbb{N}}$ in this case. The ambiguity for k_1 is resolved by demanding that $k_1=0$ for $[\pi,\mathbf{1}_m]$, unless $\pi=\mathbf{1}_m$, in which case we put $k_1=1$. Thus, if there should appear some $NC(1)$ in our following procedure, we will throw it away. This is equivalent to the following explicit definition of k_1 for a general segment $l\pi,\sigma]$: it is given as the number of blocks of σ which are also blocks of π.
Let us now consider $[\pi,\mathbf{1}_m]$ with $\pi=\{V_1,\dots,V_p\}\in NC(m)$ and put $T:=S_{(m)}=(1,\dots,m)$. If possible, choose a V_i and two neighbouring elements $k<l$ in V_i, i.e. $V_i=(\dots k,l\dots)$, such that $k+1\neq l$. Put

$$T_0:=T\backslash[k+1,l-1],\qquad T_1:=\{k\}\cup[k+1,l-1].$$

Then

$$\pi_0:=\pi\cap T_0$$

is a non-crossing partition of T_0 and

$$\pi_1:=\{(k)\}\cup\big(\pi\cap[k+1,l-1]\big)$$

is a non-crossing partition of T_1 and we have

$$[\pi,\mathbf{1}_m]\cong[\pi_0,\mathbf{1}_{T_0}]\times[\pi_1,\mathbf{1}_{T_1}].$$

This decomposition reflects the non-crossing character of our partitions, which implies that the block V_i separates the points lying between k and l from the points lying outside this interval.
If it should happen that one of the two factors is isomorphic to $NC(1)$, then we will forget this factor.
Now we can repeat the above procedure (for each of the factors) until we end up with some $[\sigma,\mathbf{1}_{\tilde m}]$, where we do not find some V_i with the above properties any more. But in this case $[\sigma,\mathbf{1}_{\tilde m}]\cong NC(|\sigma|)$.
One should note that the resulting sequence $(k_n)_{n\in\mathbb{N}}$ is independent of the order of choosing the V_i or the neighbouring k,l. $\square$

1.3.2. Definition. 1) The canonical sequence $(k_n)_{n\in\mathbb{N}}$ corresponding to some segment $[\pi,\sigma]$ is called the *class of* $[\pi,\sigma]$.
2) The *class of* $\pi\in NC(n)$ is defined as the class of $[\mathbf{0}_n,\pi]$.

Note from the foregoing proof that the class of π coincides with the corresponding object if we consider π as an element of the lattice of all partitions (cf. [Rot2]), namely the class of π is $(k_n)_{n\in\mathbb{N}}$, where k_n is equal to the number of blocks in π which contain exactly n elements. In [Kre] the class of π was called 'type of π'. Contrary to the situation for segments of the form $[\mathbf{0}_n,\pi]$, for more general segments $[\pi,\sigma]$ the notion of class gives something different in the lattice of all partitions compared to the lattice of non-crossing partitions, reflecting the more complicated structure of the latter lattice.

1.3.3. EXAMPLE. Consider

$$[\pi,\sigma] = [\{(1,5,8),(2,4),(3),(6),(7),(9,10)\},\{(1,2,3,4,5,6,7,8),(9,10)\}]$$
$$\cong [\{(1,5,8),(2,4),(3),(6),(7)\},\mathbf{1}_8] \times [\{(9,10)\},\{(9,10)\}],$$

which shows that $k_1 = 1$. Furthermore

$$\begin{aligned}
[\{(1,5,8),(2,4),(3),(6),(7)\},\mathbf{1}_8] &\cong \\
&\cong [\{(1,5,8),(2,4),(6),(7)\},\mathbf{1}_{\{1,2,4,5,6,7,8\}}] \times NC(2) \\
&\cong [\{(1,5,8),(6),(7)\},\mathbf{1}_{\{1,5,6,7,8\}}] \times NC(2) \times NC(2) \\
&\cong [\{(1,5,8)\},\mathbf{1}_{\{1,5,8\}}] \times NC(3) \times NC(2) \times NC(2) \\
&\cong NC(1) \times NC(2)^2 \times NC(3) \\
&\cong NC(2)^2 \times NC(3).
\end{aligned}$$

Since $k_1 = 1$, we have finally calculated the class of $[\pi,\sigma]$ as $(1,2,1,0,0,\dots)$.

1.3.4. DEFINITION. 1) A function $\theta \in \mathbf{I}_2$ is called *multiplicative*, if there exists a sequence of constants $(f_n)_{n\in\mathbb{N}}$ ($f_n \in \mathbb{C}$) such that

$$\theta(\pi,\sigma) = \prod_{n=1}^{\infty} f_n^{k_n} \qquad (\text{where } 0^0 := 1),$$

if the class of $[\pi,\sigma]$ is $(k_n)_{n\in\mathbb{N}}$.
2) A function $\Theta \in \mathbf{I}$ is called *multiplicative*, if there exists a sequence of constants $(f_n)_{n\in\mathbb{N}}$ ($f_n \in \mathbb{C}$) such that

$$\Theta(\pi) = \prod_{n=1}^{\infty} f_n^{k_n} \qquad (\text{where } 0^0 := 1),$$

if the class of π is $(k_n)_{n\in\mathbb{N}}$. In this case, we write also $\Theta = \hat{f} = (f_n)$.

With the notations as in our definition we have that θ is the unique multiplicative extension of Θ and Θ is the restriction of θ to segments of the form $[\mathbf{0}_n,\pi]$. Of course, the restriction of a multiplicative function is multiplicative, too.

1.3.5. REMARKS. 1) Our special functions ζ, δ, and μ are all multiplicative: ζ is determined by the sequence $(1,1,1,\dots)$ and δ is given by $(1,0,0,\dots)$. The sequence $(\mu(\mathbf{0}_n,\mathbf{1}_n))_{n\in\mathbb{N}}$ for the Möbius function will be determined in Corollary 2.3.10.
2) For $\Theta \in \mathbf{I}$, the terminology 'multiplicative' has also the following quite instructive explanation: We can write $\pi \in NC(n)$ in the form $\pi = \pi_1 \cup \mathbf{1}_V$, where V is an interval in $(1,\dots,n)$ and $\pi_1 \in NC(S_{(n)}\backslash V)$. Then

$$\Theta(\pi) = \Theta(\pi_1 \cup \mathbf{1}_V) = \Theta(\pi_1)\Theta(\mathbf{1}_V) = \Theta(\pi_1) f_{|V|},$$

which may be iterated to give finally

$$\Theta(\pi) = f_{|V_1|}\cdots f_{|V_p|} \qquad \text{for } \pi = \{V_1,\dots,V_p\}.$$

This form of multiplicativity will be generalized in the next chapter to operator-valued functions.
3) Note that obviously

$$[\vec{\pi}, \vec{\sigma}] \cong [\pi, \sigma],$$

and thus

$$\theta(\vec{\pi}, \vec{\sigma}) = \theta(\pi, \sigma) \qquad \text{for } \theta \in \mathbf{I}_2$$

and

$$\Theta(\vec{\pi}) = \Theta(\pi) \qquad \text{for } \Theta \in \mathbf{I}.$$

1.3.6. Proposition. *1) The convolution of two multiplicative functions in $\mathbf{I}_2$ is multiplicative.*
2) If $\Theta \in \mathbf{I}$ and $\eta \in \mathbf{I}_2$ are multiplicative, then $\Theta \star \eta \in \mathbf{I}$ is multiplicative, too.

Proof. 1) Let

$$[\pi, \sigma] = \prod_{n \in \mathbb{N}} [\mathbf{0}_n, \mathbf{1}_n]^{k_n}.$$

Then we have for $\theta, \eta \in \mathbf{I}_2$

$$\begin{aligned}(\theta \star \eta)(\pi, \sigma) &= \sum_{\pi \leq \nu \leq \sigma} \theta(\pi, \nu)\eta(\nu, \sigma) \\ &= \prod_{n \in \mathbb{N}} \Big(\sum_{\nu_n \in NC(n)} \theta(\mathbf{0}_n, \nu_n)\eta(\nu_n, \mathbf{1}_n)\Big)^{k_n} \\ &= \prod_{n \in \mathbb{N}} g_n^{k_n}\end{aligned}$$

with

$$g_n := \sum_{\nu_n \in NC(n)} \theta(\mathbf{0}_n, \nu_n)\eta(\nu_n, \mathbf{1}_n) = (\theta \star \eta)(\mathbf{0}_n, \mathbf{1}_n).$$

2) Let $\theta \in \mathbf{I}_2$ be the multiplicative extension of Θ. Then $\Theta \star \eta \in \mathbf{I}$ is the restriction of $\theta \star \eta \in \mathbf{I}_2$. $\square$